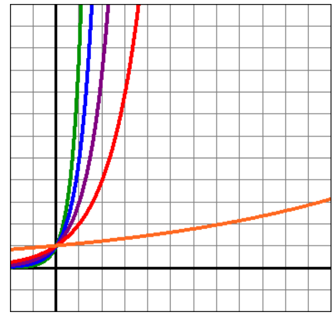
Calculus 130, section 4.4-4.5 Derivatives of Exponential & Logarithm Functions

notes by Tim Pilachowski

You'll need everything we covered in sections 2.1 – 2.3, especially exponential and logarithm properties.



Functions with the basic form $y = b^x$ are actually a family of functions. Consider the functions $y = 10^x$, $y = 5^x$, $y = 3^x$, $y = 2^x$, $y = 1.1^x$ pictured in the graph to the right. Note first the similarities: $b^0 = 1$ for all values of $b \neq 0$, so (0, 1) makes a good reference point. Each of the basic exponential functions has a horizontal asymptote $y = 0$. The graphs also have similar shape—the major difference is slope of the curve at specific values of x. Note that at $x = 0$ slope of $y = 10^x$ is steepest; slope of $y = 1.1^x$ is most shallow. Our determination of first derivative will have to reflect this.

Recall that our limit definition for the first derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, is akin to considering a series of secant lines where the second point comes ever closer to a fixed point $(x, f(x))$.

First, consider $y = 5^x$ at $x = 0$ where $y = 1$. The slope of the secant line connecting (0, 1) to another point on the curve is given by the formula $\frac{5^{0+h} - 5^0}{h} = \frac{5^h - 1}{h}$. As our second point approaches (0, 1), the slope of the secant line approaches the slope of the tangent line, i.e slope of the tangent = $\lim_{h \to 0} \frac{5^h - 1}{h}$. The table below provides results from successively smaller values of h.

$5^0 =$	$h =$	$5^h \cong$	$5^h - 1 \cong$	$\frac{5^h - 1}{h} \cong$
1	1	5	4	4
1	0.1	1.174618943	0.174618943	1.746189431
1	0.01	1.016224591	0.016224591	1.622459127
1	0.001	1.001610734	0.001610734	1.610733753
1	0.0001	1.000160957	0.000160957	1.609567434
1	0.00001	1.000016095	0.000016095	1.609450864
1	0.000001	1.000001609	0.000001609	1.609439208
1	0.0000001	1.000000161	0.000000161	1.609438043
1	0.00000001	1.000000016	0.000000016	1.609437916

The slope of the tangent to $y = 5^x$ at $x = 0$, and therefore the first derivative of $y = 5^x$ at $x = 0$, is approximately 1.61. This is an estimate and is not exact–with some higher-powered mathematics, we would identify the exact expression of the first derivative of $y = 5^x$ at $x = 0$ as ln (5).

Next we move to an arbitrary point $(x, 5^x)$ on the graph of $y = 5^x$. The slope of the secant line connecting $(x, 5^x)$ to another point on the curve is given by the formula $\frac{5^{x+h} - 5^x}{h}$, which can be simplified using the properties of exponents: $\frac{5^{x+h} - 5^x}{h} = \frac{5^x * 5^h - 5^x}{h} = \frac{5^x (5^h - 1)}{h}$. As our second point approaches $(x, 5^x)$, the slope of the secant line approaches the slope of the tangent line, i.e slope of the tangent =

$$\lim_{h \to 0} \frac{5^x (5^h - 1)}{h} = \left[\lim_{h \to 0} \frac{(5^h - 1)}{h} \right] * 5^x = \left[\left. (5^x)' \right|_{x=0} \right] * 5^x = \left[\left. \frac{d}{dx} (5^x) \right|_{x=0} \right] * 5^x \cong \ln(5) * 5^x .$$

If we rigorously followed a similar process for a generic function $y = b^x$, we would get $y' = \ln(b) * b^x$. When considering the natural exponential function $f(x) = e^x$, this gives us $f'(x) = \ln(e) * e^x = e^x$! In short, the natural exponential function, $y = e^x$, is special because, unlike other functions, it is its own derivative. This property makes it not only very interesting, but also very useful.

Example A: Find the first derivative of $f(x) = \dfrac{x^3}{e^x}$ and solve $f'(x) = 0$. *Answers*: $\dfrac{3x^2 - x^3}{e^x}$; $x = 0, 3$

Example B: Given $h(x) = e^{x^2 - x}$, find the first derivative. *Answer*: $\left(e^{x^2 - x}\right)(2x - 1)$

Example C: Let $g(t) = e^{2t} + e^{-2t}$. Determine where the graph of g has horizontal tangents. *Answer*: $t = 0$

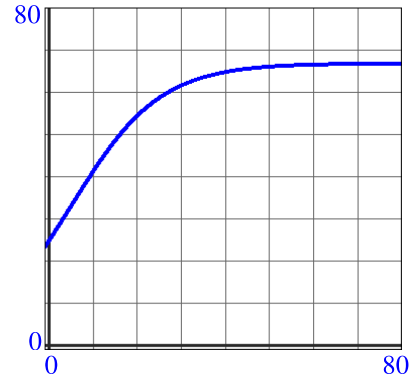
Example D: The accumulated amount of an investment of $100 with 5% annual interest compounded continuously for t years is given by the formula $A = 100e^{0.05t}$. Find and interpret the first derivative. *Answer*: $5e^{0.05t}$

From Examples C and D we can make a generic observation: Given an exponential growth/decay function $y = Ce^{kx}$, the derivative will be $y' = C * ke^{kx} = k * Ce^{kx} = ky$.

Warning! Danger! Be careful! This observation applies only to basic exponential growth and decay! It will not apply to other functions involving exponentials.

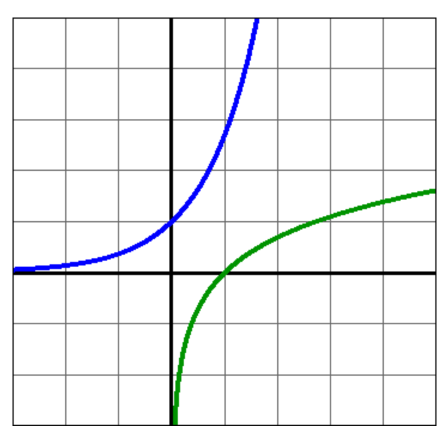
Example E: The exponential growth model $y = Ce^{kx}$ applied to populations of people or animals has a serious flaw: In the real world the number that can survive is limited by the amount of space and the number of resources available. A *logistic growth curve* is more appropriate for long-term applications.

The population of deer in a wildlife preserve is modeled by $P(t) = \dfrac{200}{3 + 5e^{-0.1t}}$. a) What was the number of deer at the beginning? b) What is the theoretical "upper limit" according to this model? c) How quickly is the population growing after 10 years? *Answers*: 25 deer; 66-67 deer; $\dfrac{100}{e\left(3+\frac{5}{e}\right)^2} \approx 1.6$ deer per year



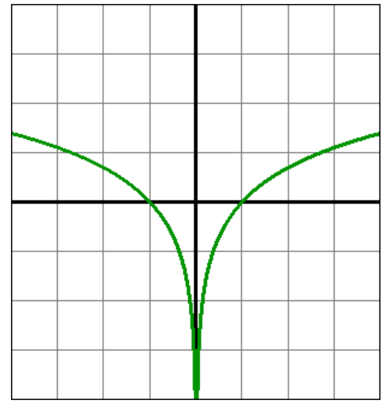
Example F: Given $f(x) = \pi^x$, find $f'(x)$. *Answer*: $(\ln \pi) * \pi^x$

The *natural logarithm* function, $y = \ln(x)$, is the inverse of the natural exponential function, $y = e^x$. By finding the first derivative, we can determine that the slope of $y = e^x$ is 1 at the point (0, 1), i.e. $\left.\dfrac{d}{dx}\left(e^x\right)\right|_{x=0} = 1$. By symmetry, the slope of $y = \ln(x)$ should also be 1 at the point (1, 0), i.e. $\left.\dfrac{d}{dx}(\ln x)\right|_{x=1} = 1$. Also recall that $y = \ln(x)$ is increasing over its entire domain The formula we use for the derivative of ln(x) must meet these conditions.



Finding a derivative formula for ln(x) is actually quite simple. First note that since $e^{\ln x} = x$, then $\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}(x) = 1$. By the chain rule, $\frac{d}{dx}\left(e^{\ln x}\right) = e^{\ln x} * \frac{d}{dx}(\ln x) = x * \frac{d}{dx}(\ln x) = 1 \Rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$.

Note that $\left.\frac{d}{dx}(\ln x)\right|_{x=1} = \left.\frac{1}{x}\right|_{x=1} = 1$. Also $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$ and $\frac{d^2}{dx^2}(\ln x) = -\frac{1}{x^2} < 0$ for all x in the domain of ln(x). In other words, all of the necessary conditions listed above have been met.



Also note that, since the domain of $f(x) = \ln x$ is $(0, \infty)$, the domain of $f'(x) = \frac{1}{x}$ is also $(0, \infty)$. What if we were to consider $g(x) = \ln|x|$ which has a domain $= (-\infty, 0) \cup (0, \infty)$. Using symmetry of the graph of g across the y-axis, we could show that it is also true that $g'(x) = \frac{1}{x}$ for all values of x in the domain $(-\infty, 0) \cup (0, \infty)$.

Example G: Given $h(x) = x^3 * \ln|x|$ find the first derivative.

Answers: $x^2(1 + 3\ln|x|)$

Example H: Given $f(x) = \frac{x^3}{\ln|x|}$ find the first derivative. *Answer*: $\frac{x^2(3\ln|x| - 1)}{[\ln|x|]^2}$

Example I: Given $g(x) = \frac{\ln|x|}{x^3}$ find the first derivative. *Answer*: $\frac{1 - 3\ln|x|}{x^4}$

Carefully note the placement of coefficients when finding derivatives.

constant multiple rule

$m(x) = k\ln(x)$

chain rule

$n(x) = \ln(k\,x)$

Example J: Give $h(x) = \log_{\pi}(x^3)$, find the first derivative. *Answer*: $\dfrac{3}{\ln(\pi)} * \dfrac{1}{x}$

When using the chain rule, it is extremely important to correctly identify the "outside" and "inside" functions. Check that your composition is set up correctly.

Example K: Give $h(x) = [\log_{\pi} x]^5$, find the first derivative. *Answer*: $\dfrac{5(\ln x)^4}{[\ln \pi]^5\,x}$