Calculus 130, section 5.3 Second Derivative, Concavity, Points of Inflection

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Reminder: You will not be able to use a graphing calculator on tests!

To review what we already have, the following statements are mathematically equivalent:

a) Find the slope of the line tangent to the graph of f at a point (x, y).

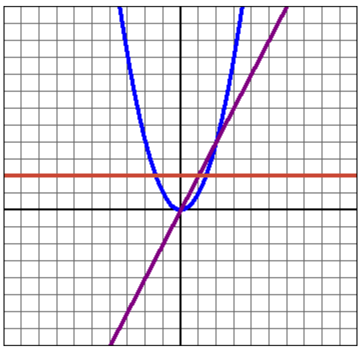
b) Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ c) Find the first derivative of $f(x)$. d) Find $f'(x) = \frac{dy}{dx} = D_x$.

Recall, however, that the first derivative is itself a function, which has its own domain and its own graph. Since it is a function, it also has its own derivative. Given a function f, we can calculate the first derivative f' or $\dfrac{dy}{dx}$.

We can then calculate the derivative of f', i.e. the second derivative of f, symbolically $f'' = \dfrac{d^2y}{dx^2} = D_x^2[f]$.

<u>Important note</u>: Just like $\dfrac{dy}{dx}$ is *not* a fraction, but is a notation for the first derivative, $\dfrac{d^2y}{dx^2}$ is also not a fraction but a notation. *There is no multiplication involved!* Rather, you need to interpret it this way:

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ which means "the derivative of $\dfrac{dy}{dx}$", the derivative of a derivative.



Examples A. Consider the graph of $y = x^2$ pictured to the left along with its derivatives $y' = 2x$ and $y'' = 2$.

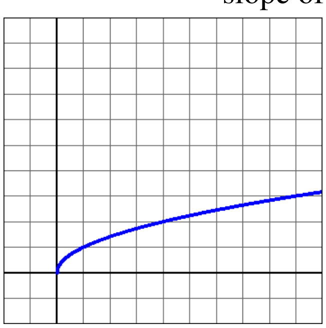
Interval	$y = x^2$ is…	$y' = 2x$ is…	$y'' = 2$ is…

There is a connection between the concavity of a function and its second derivative. The second derivative is the slope of the first derivative, and tells us how the first derivative is changing, i.e. how the slope of the function is itself changing. In the graph of $y = x^2$ above, the slope (first derivative) is negative on the interval $-\infty < x < 0$. Note that the slope of the parabola is becoming less steep (more shallow) as x approaches 0. Another way to say the same thing is that the slope of the parabola (first derivative), while still negative, is becoming less negative as x approaches 0, until the curve hits $x = 0$, at which point the slope of the parabola (first derivative) equals 0. On the interval $0 < x < \infty$ the slope of the parabola (first derivative) is positive. Note, too, that the slope of the parabola is becoming steeper (i.e. the first derivative is becoming ever-larger positive numbers) as x approaches ∞. The slope of the curve = the first derivative is progressing in this way:

very negative < less negative < zero < small positive < large positive

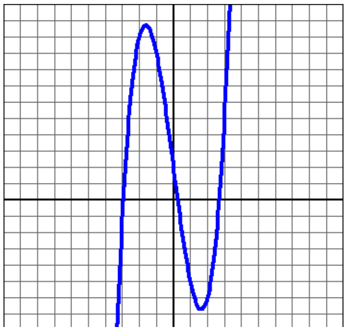
slope of curve is always increasing = first derivative is always increasing

= slope of first derivative is always positive = second derivative is always positive



Now consider $y = \sqrt{x} = x^{1/2}$.

Example B: Let's go back to $f(x) = x^3 - 8x + 2$ and take a closer look at the curve, pictured to the left below.



first derivative: $f'(x) = 3x^2 - 8$

second derivative:

critical numbers:

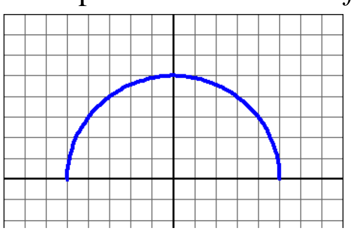
critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:

Example C: The function $f(x) = \sqrt{25 - x^2}$ has a limited domain, $-5 \le x \le 5$, and range, $0 \le y \le 5$.



first derivative: $f'(x) = \dfrac{-x}{\sqrt{25 - x^2}}$

second derivative:

critical numbers:

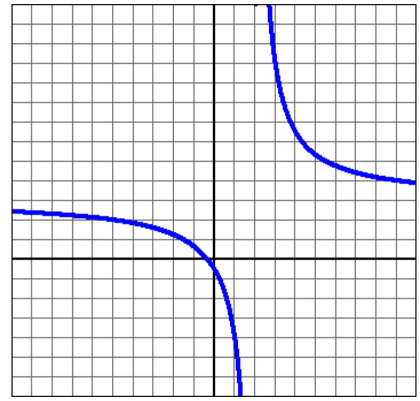
critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:

Example D: Consider the function $f(x)=\dfrac{3x+1}{x-2}$.



first derivative: $f'(x)=\dfrac{-7}{(x-2)^2}$

second derivative:

critical numbers:

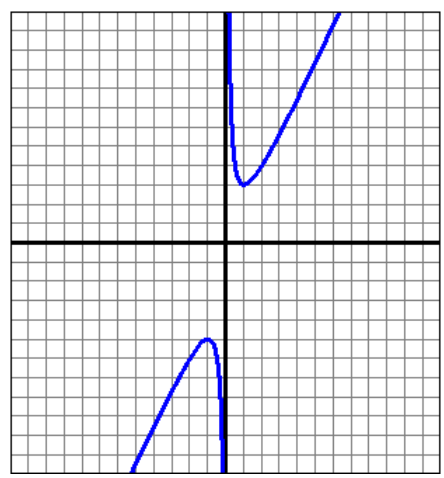
critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:

Example E: Consider the function $f(x)=2x+\dfrac{2}{x}-1=2x+2x^{-1}-1$.



first derivative: $f'(x)=2-\dfrac{2}{x^2}$

second derivative:

critical numbers:

critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:

Example F: Consider the function $f(x) = \dfrac{x^3}{e^x}$.

first derivative: $f'(x) = \dfrac{3x^2 - x^3}{e^x}$

second derivative:

critical numbers:

critical points:

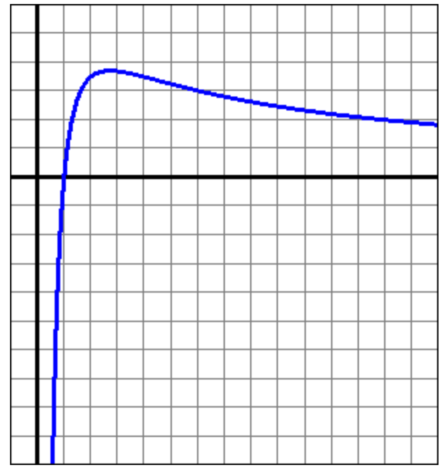
interval(s) concave up:

interval(s) concave down:

point(s) of inflection:

Example G: The concentration of a drug in the bloodstream t hours after injection into a muscle is given by $c(t) = 9\left(e^{-0.3t} - e^{-3t}\right)$ units. Find the time at which the ***rate of absorption*** of the drug in the bloodstream is at its maximum.

Example H: Consider the function $f(x) = \frac{10 \ln x}{x}$.



first derivative: $f'(x) = \dfrac{10 - 10 \ln x}{x^2}$

second derivative:

critical numbers:

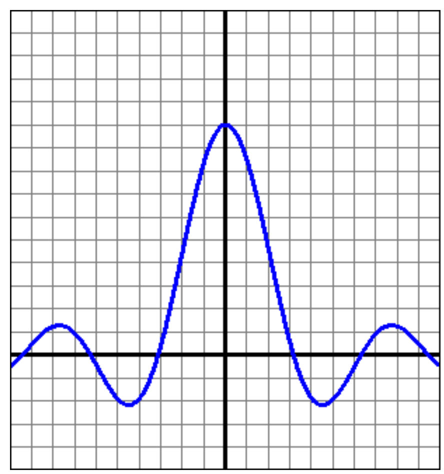
critical points:

interval(s) concave up:

interval(s) concave down:

point(s) of inflection:

Example I: Consider the function $f(x) = \frac{10 \sin x}{x}$. Determine the concavity of the graph at the values of x given below.

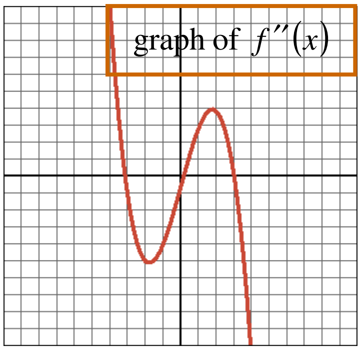


first derivative: $f'(x) = \dfrac{10(x \cos x - \sin x)}{x^2}$

second derivative:

concavity at $x = -\pi$:

concavity at $x = \dfrac{\pi}{2}$:



Example J: Without knowing the function itself, describe the behavior of its graph only using information provided by its second derivative. The graph to the left is a graph of $f''(x)$.

Since $f''(x) = 0$ at $x = -3$, $x \cong 0.1$, and $x = 3$, we'll look in those places for points of inflection. We can determine whether *f* is concave up or down by determining where f'' is positive or negative.

interval	$x < -3$	$x = -3$	$-3 < x < 0.1$	$x \cong 0.1$	$0.1 < x < 3$	$x = 3$	$3 < x$
value of f''							
f is concave…							

interval(s) concave up:

interval(s) concave down:

locations of points of inflection:

Using this information, along with information from Lecture 5.1-5.2, we can draw a possible graph for *f*, which may look something like this: