

Calculus 130, section 7.1 Antiderivatives

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In mathematics we often go backwards. Subtraction is the “backwards” of addition: $2 + 3 = 5 \Rightarrow 5 - 3 = 2$.

Division is the “backwards” of multiplication: $4 * 5 = 20 \Rightarrow 20 \div 5 = 4$. Solving is the “backwards” of calculation: $3(2 + 5) = 30 \Rightarrow 3(x + 5) = 30$.

An integral, or antiderivative, is the “backwards” of finding the first derivative. In the real world, we may know something about the rate of change from experiment or observation, and want to work our way backwards to find the equation that describes a phenomenon.

Example A: Given a function $f(x) = 5x^4$ find a function $F(x)$ such that $F'(x) = f(x)$. *Answer:* $F(x) = x^5 + C$

Example A extended: Given a function $f(x) = 5x^4$ find a function $F(x)$ such that $F'(x) = f(x)$ and $F(1) = 8$.

Answer: $F(x) = x^5 + 7$

In the example above, the question was phrased, “Find a function $F(x)$ such that $F'(x) = f(x)$.” There are four other equivalent ways to ask the same thing:

Find the antiderivative of $f(x)$.

Integrate $f(x)$.

Find the integral of $f(x)$.

Find $\int f(x) dx$.

Example B: Find the antiderivative of $f(x) = x^4$. *Answer:* $\frac{1}{5}x^5 + C$

From this example, we can generalize the process for integrating power functions:

$$\int x^r dx = \frac{1}{r+1} x^{r+1} + C, \quad r \neq -1.$$

Note the restriction on r . We have to avoid a 0 in the denominator since division by 0 is undefined.

Example C: Evaluate $\int \frac{1}{\sqrt{x}} dx$. Answer: $2\sqrt{x} + C$

Example D: Integrate $f(x) = x^{-1}$. Answer: $\ln|x| + C$

Example D addendum: Evaluate $\int \frac{5}{x} dx$ and $\int \frac{1}{6x} dx$. Answers: $5\ln|x| + C$, $\frac{1}{6}\ln|x| + C$

Just like differentiation, integration has a constant multiple rule.

$$\frac{d}{dx}[a * f(x)] = a * \frac{d}{dx}[f(x)] \quad \int a * f(x) dx = a * \int f(x) dx$$

Now is as good a time as any to point out the “ dx ” part of the integral $\int f(x) dx$. It is a necessary part of any integral, since we are finding the antiderivative of $f(x) = \frac{d}{dx}[F(x)] \rightarrow F(x) = \int f(x) dx$.

Example E: Find the integral of $f(x) = e^{-2x}$. Answer: $-\frac{1}{2}e^{-2x} + C$

From this example, we can generalize the process for integrating exponential growth/decay functions:

$$\int e^{kx} dx = \frac{1}{k}e^{kx} + C, \quad k \neq 0.$$

Note the restriction on k . We have to avoid a 0 in the denominator since division by 0 is undefined.

Example F: Find the antiderivative of $f(x) = e^0$. *Answer:* $x + C$

Note that in this example, as in all the others, we can easily check our answer by finding its derivative:

$\frac{d}{dx}(x + C) = 1$, which is correct. Checking your antiderivative by finding the derivative is a good habit to develop.

IMPORTANT NOTE:

Example F addendum: Find $\int 0 \, dx$. *Answer:* $y = C$

Example G: Find $\int (3x^{-6} - 2e^{5x} + 4x^{-1} - 7) \, dx = \int 3x^{-6} - 2e^{5x} + 4x^{-1} - 7 \, dx$.

While the parentheses surrounding the function may be helpful they are technically not necessary. The integral “begins” at the \int symbol and “ends” at the dx . Everything in-between is included.

Answer: $-\frac{3}{5}x^{-5} - \frac{2}{5}e^{5x} + 4\ln|x| - 7x + C$.

Recall the sum rule for derivatives? The same rule applies to antiderivatives.

So far in this class, we have spent chapters 1 through 6 beginning with a function that represents an *amount*. The derivative (= slope of the curve) gives us a *rate of change*.

In chapter 7, we are beginning with a *rate of change* and can use the integral to determine an *amount*.

In applications, it is first important to determine the question—to identify what sort of answer you are looking for: Is it an *amount* or is it a *rate of change*? Next identify the nature of the information you have: Is it an *amount* or is it a *rate of change*?

Example H: An object falls from a high-altitude balloon (4000 feet above the ground) with a speed of $32t$ feet per second. Find an equation to express the height h of the object at time t .

“Height of the object” asks for a distance, which is an *amount*. The velocity is a *rate of change* of distance with respect to time: $v(t)$ in $\frac{\text{feet}}{\text{second}}$ is $\frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta h}{\Delta t}$.

Answer: $h(t) = -16t^2 + 4000$ (label?)

Example I: The rate at which gasoline was consumed in the U.S. was approximately $r(t) = 0.075t + 1.7$ billion gallons per year from 1964 ($t = 0$) to 1976. Find a function $f(t)$ to express the number of gallons of gasoline consumed in a given year.

The equation given is a *rate of change* of consumption. The “number of gallons” asked for is an *amount*.

Answer: $f(t) = 0.0375t^2 + 1.7t + C$ (label?)