Calculus 130, section 7.2 Integration by Substitution

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From section 7.1, we have five ways to ask the same question:

Find a function $F(x)$ such that $F'(x) = f(x)$.

Find the antiderivative of $f(x)$. Integrate $f(x)$.

Find the integral of $f(x)$. Find $\int f(x)\,dx$.

We also have two generic integration properties (or rules):
the constant multiple rule and the sum rule.

We also have three anti-derivative rules:

$$\text{power functions, } \int x^r\ dx = \frac{1}{r+1}x^{r+1} + C, \quad r \neq -1$$

$$\text{exponential functions, } \int e^{kx}\ dx = \frac{1}{k}e^{kx} + C, \quad k \neq 0$$

$$\text{recognizing the antiderivative of } f(x) = x^{-1}, \ \int \frac{1}{x}\,dx = \ln|x| + C, \quad x \neq 0.$$

Note that finding the antiderivative of a constant is actually an application of the power rule, although it may be easier just to recognize and remember, paying attention to the differential *d**, "with respect to" variable:

$$\int 5\,dx = \int 5x^0\ dx = 5 * \left(x^{0+1} \div 1\right) + C = 5x + C$$

$$\int \pi\ dy = \pi y + C \qquad \int e^7\ dt = e^7 t + C \qquad \int \ln(9)\,dw = w\ln(9) + C.$$

Now we begin to address integrals which are not as easy as finding the antiderivative. The first method is called *integration by substitution*, and is like a "chain rule for derivatives" in reverse. (This is the only additional method of integration we'll cover in Math 130. There are others addressed in Math 131.)

Recall that, by the chain rule, $\frac{d}{dx}F[g(x)] = F'[g(x)] * g'(x)$. For an integral that we can recognize as $\int F'[g(x)] * g'(x)\,dx$, we can integrate our way back to $F[g(x)] + C$. The hard part is the recognition of this form.

Example A: Find $\int \frac{2\ln x}{x}\,dx$. *answer*: $(\ln x)^2 + C$

Example A extended: Find $\int \frac{2(1+\ln x)}{x}\,dx$. *answer*: $(1+\ln x)^2 + C$

Based on the knowledge of derivatives covered so far, besides the substitution above there are some general forms of integrals to look for:

$$\int u^n \; du \qquad\qquad \int e^u \; du \qquad\qquad \int \frac{1}{u} \; du$$

Example B: Evaluate $\int \frac{3}{\sqrt{3x+1}}\, dx$. *answer*: $2(3x+1)^{1/2} + C$

Example B extended: Evaluate $\int \frac{6x}{\sqrt{3x^2+1}}\, dx$. *answer*: $2\left(3x^2+1\right)^{1/2} + C$

Example B again: Evaluate $\int \frac{9x^2+2}{\sqrt{3x^3+2x+1}}\, dx$. *answer*: $2\left(3x^3+2x+1\right)^{1/2} + C$

Example C: $\int x\left(x^2+3\right)^7 \, dx$. *answer*: $\frac{1}{16}\left(x^2+3\right)^8 + C$

Example D: $\int 6x\,e^{x^2-1}\,dx$. *answer*: $3e^{x^2-1}+C$

Example E: $\int \frac{x}{e^{x^2}}\,dx$. *answer*: $-\frac{1}{2}e^{-x^2}+C$

Example F: $\int \frac{x^2}{x^3+8}\,dx$. *answer*: $\frac{1}{3}\ln\left|x^3+8\right|+C$

Hint: When using substitution in an integral involving polynomials, it is usually most productive to let u = (polynomial with the higher exponent). When using substitution in an integral involving ln(*a function)*, it is usually most productive to let u = ln(*a function)*. When using substitution in an integral involving $e^{\text{(exponent function)}}$, it is usually most productive to let u = (exponent function).

Example G: $\int xe^x\,dx$.

We have no product rule for integrals. If we try letting $u = x$, we get $\int ue^u\,du$, which is no help at all. If we then try letting $u = e^x, \quad du = e^x\,dx$, we'd have $\int xu\,du$, which we cannot integrate—it does not have matching variables!

In Lecture 7.5 we'll bring the trigonometric functions into the substitution method.