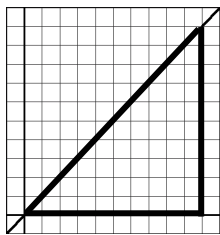


# Calculus 130, section 7.3 Area Under a Curve & Riemann Sums

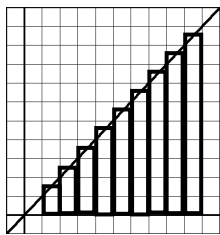
notes by Tim Pilachowski



Consider the function  $f(x) = x$  on the interval  $[0, 10]$ . With the  $x$ -axis (the horizontal line  $y = 0$ ) and the vertical line  $x = 10$ ,  $f$  forms a triangle. We could find the area of the triangle by counting squares. (There are 45 full squares and 10 half-squares for a total of 50.) of them.) An easier method would be to use knowledge of geometry to calculate the area of that triangle, which is also, by the way, the “area under the curve”. Putting the correct values into the formula  $A = \frac{1}{2}bh$  we get

$$\text{area of triangle} = \text{area under the curve} = \frac{1}{2} * 10 * 10 = 50.$$

This scenario is fairly easy, because the function  $f(x) = x$  is a line, and the “area under the curve” of  $f$  forms a well-known geometric shape. What happens when the curve is not linear but actually curves? Since there is no geometric formula for irregularly-shaped spaces, we’ll need a way to approximate the area under the curve.



Suppose that we form a series of rectangles under  $f(x) = x$  on the interval  $[0, 10]$  and use those to approximate the area under the curve. If we draw in 10 rectangles of width = 1, and put the midpoint of the top of each rectangle on the line  $f(x) = x$ , the sum of the areas of the rectangles = approximation of the area under the curve =

$$1(0.5) + 1(1.5) + 1(2.5) + 1(3.5) + 1(4.5) + 1(5.5) + 1(6.5) + 1(7.5) + 1(8.5) + 1(9.5) = 50.$$

In more general terms, the interval  $[a, b]$  was split up into  $n$  subintervals, called *partitions*, of

$$\text{width} = \frac{b-a}{n} = \Delta x.$$

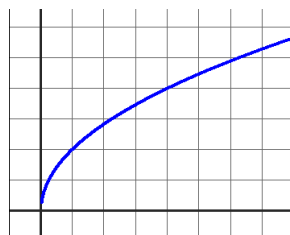
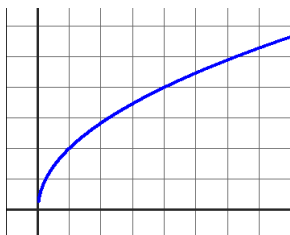
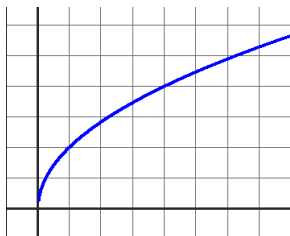
The height of each rectangle is a  $y$ -value,  $f(x)$  evaluated at the midpoint of the partition.

$$\text{Area under the curve} \cong f(x_1) * \Delta x + f(x_2) * \Delta x + f(x_3) * \Delta x + \dots + f(x_n) * \Delta x$$

$$= [f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n)] * \Delta x = \sum_{i=1}^n f(x_i) * \Delta x$$

This formula is called a *Riemann* sum, and provides an approximation to the area under the curve for functions that are non-negative and continuous. In your homework exercises you will be asked to use this midpoint version of a Riemann sum, as well as left and right endpoints, along with the average of left and right endpoint sums.

Example A: Approximate the area under the curve  $y = 2\sqrt{x}$  on the interval  $2 \leq x \leq 7$  using five partitions and left endpoint sum, right endpoint sum, average of left and right endpoint sums, and midpoint sum.

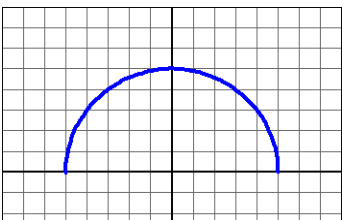


Example A extended: Repeat the approximation process using 10 partitions (left endpoint sum, right endpoint sum, average of left and right endpoint sums, and midpoint sum).

The exact value for the area under the curve  $y = 2\sqrt{x}$  on the interval  $2 \leq x \leq 7$  is  $\frac{4}{3}(\sqrt{7^3} - \sqrt{2^3})$  which is approximately 20.92244274. Just as increasing the number of partitions brought us closer to the true value for the area under the curve  $y = 2\sqrt{x}$ , it is reasonable to suppose that, in general, for any function, increasing the number of partitions will provide an increasingly better approximation to the area under the curve. If we look at the limit of a Riemann sum as the number of partitions  $n$  approaches  $\infty$ , we get a *definite integral*.

Given a closed interval  $[a, b]$  and  $\Delta x = \frac{b-a}{n}$ , then  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) * \Delta x = \int_a^b f(x) dx$ .

Example B: Find the exact values of  $\int_{-5}^5 \sqrt{25-x^2} dx$  and  $\int_0^5 \sqrt{25-x^2} dx$  using formulas from geometry.



Now comes an important question: Why would we be interested in the area under a curve? What is the connection between area under the curve and integration?

Consider a velocity function  $v(t)$ . When  $v(t)$  is constant, it is not difficult to see that the formula “distance = rate of speed \* time” is the area of the rectangle formed on the graph to the right, i.e. “distance = area under the curve  $v(t)$ ”.

When  $v(t)$  is changing, the area of the rectangles formed by our partitions gives us “average rate of speed on the partition \* time = area under the curve = distance”.

We’ve already determined that “velocity = rate of change of distance = first derivative of distance”.

This relationship turned backwards is “distance = antiderivative of velocity”.

Substituting from the observations above we can conclude “area under the curve  $v(t)$  = antiderivative of  $v(t)$  = integral of  $v(t)$ ”.

