Calculus 130, section 7.4 Definite Integrals & the Fundamental Theorem

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In Lecture 6.2 we found a Riemann sum (midpoint version) to approximate the area under a curve on an interval $a \le x \le b$, utilizing partitions and a series of rectangles with width $\frac{b-a}{n} = \Delta x$:

Area under the curve $\cong f(x_1) * \Delta x + f(x_2) * \Delta x + f(x_3) * \Delta x + \ldots f(x_n) * \Delta x$

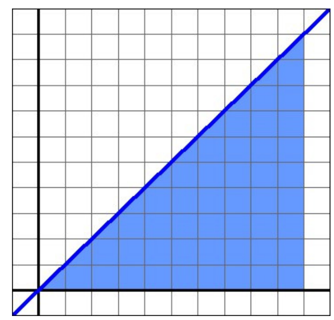
$$= [f(x_1) + f(x_2) + f(x_3) + \ldots f(x_n)] * \Delta x = \sum_{i=1}^{n} f(x_i) * \Delta x .$$

We also related the area under the curve to the antiderivative, or integral, of a function. Now we formalize the mathematics into the *definite integral* from *a* to *b*:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) * \Delta x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i) * \Delta x = \int_a^b f(x)\, dx .$$

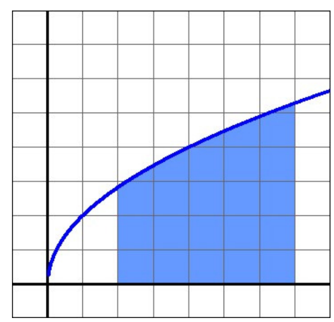
Note that the definite integral is no longer an approximation but is equal to the area under the curve.

Example A: Find the area under the curve $f(x) = x$ on the interval [0, 10].
answer: 50



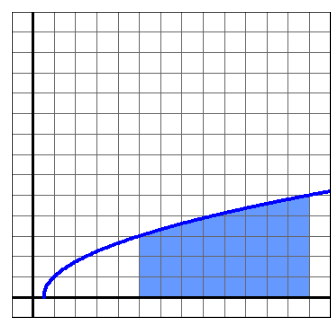
Example B: Find the area under the curve $y = 2\sqrt{x}$ on the interval $2 \le x \le 7$.

answer: $\frac{4}{3}\sqrt{7^3} - \frac{4}{3}\sqrt{2^3}$



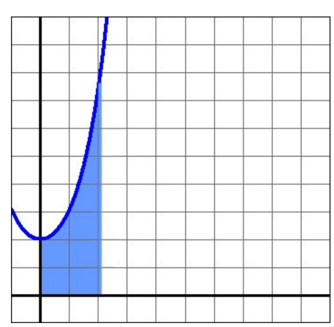
Example C: Find the area under the curve $y = \sqrt{2x-1}$ on the interval $5 \le x \le 13$.

answer: $\frac{98}{3}$



Example D: Find the area under the curve $y = e^x + e^{-x}$ on the interval $0 \le x \le \ln 8$.

answer: $\frac{63}{8}$



In all four examples above, we made use of *the Fundamental Theorem of Calculus*:
Given a function $f(x)$ which is continuous on an interval $a \le x \le b$, and given $F(x)$, an antiderivative of $f(x)$,

$$\text{then } \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Note that this theorem can make use of *any* antiderivative, so we'll generally choose the easiest version, i.e. the one without the "+ C". This works because, as in the examples above, we'd always get "+ $C - C = 0$" anyway.

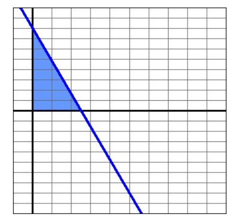
Now consider a rocket fired upward from the ground at velocity 80 feet per second. Its height as a function of time is given by the function $h(t) = -16t^2 + 80t$. The two times it at height = 0 are found by solving

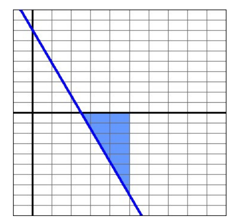
$$0 = -16t^2 + 80t = -16t(t-5) \quad \Rightarrow \quad t = 0 \text{ sec. and } t = 5 \text{ sec.}$$

The velocity of the rocket is given by the function

$$v(t) = h'(t) = -32t + 80.$$

Example E: Find and interpret "area under the curve" for the function $f(t) = -32t + 80$ on the intervals
a) $0 \le x \le 2.5$, b) $2.5 \le x \le 5$, and c) $0 \le x \le 5$.
answers: 100; – 100; 0 [100; 100; 200]





The important concept here is that while area between the curve and the x-axis which is above the x-axis is *positive*, area between the curve and the x-axis which is below the x-axis is *negative*. In an application where negative values make sense, these negative values can be useful. (For example, an object which is falling loses height.) From a geometry perspective, area is always positive, so we'll use absolute value as necessary.

Example F: Find $\int_1^{e^2} \frac{\ln x}{2x}\,dx$. *answer*: 1

Example F using the Change of Limits Rule: Find $\int_1^{e^2} \frac{\ln x}{2x}\,dx$. *answer*: 1

Example G: $\int_0^2 \frac{x}{\left(x^2+3\right)^5}\,dx$. *answer*: $-\frac{1}{8}\left[\frac{1}{7^4}-\frac{1}{3^4}\right] \cong 0.0014911482$

Example H: $\int_{-1}^{2} 5x\,e^{x^2-1}\,dx$. *answer*: $\frac{5}{2}\left[1-\frac{2}{e}+e^3\right]$

Example I: $\int_0^{10} \frac{x^2}{x^3+8}\,dx$. *answer*: $\frac{1}{3}\left[\ln 1008-\ln 8\right]=\frac{1}{3}\ln(126)$

Example J: For n = number of people infected in hundreds and t = number of days, the rate of spread of a flu infection is found to be $\frac{dn}{dt}=5+3\sqrt{t}$. How many people would we expect to become infected on days 9 to 16?
answer: 10,900 people

Example K: Blood flows through an artery fastest at the center and slowest next to the artery wall. In 1842 the French physician Poiseuille developed an equation for the rate of blood flow in an artery with radius 0.2 cm: $v(x) = 40 - 990x^2$ where v is in centimeters per second and x = distance from the center in centimeters. (The equation will be different for an artery of different length and if blood pressure is not normal.) Compare the amount of blood flowing from the center to 0.1 cm with the amount of blood flowing between 0.1 and 0.2 cm from the center. (For the correct label, think in terms of "length times width = area", in this case "cm per second times cm = cm^2 per second" which is a cross-sectional amount of blood that flows by during a time interval.)

$$\int_0^{0.1} 40 - 990x^2 \, dx = 40x - 330x^3 \Big|_0^{0.1} = \left[40(0.1) - 330(0.1)^3\right] - \left[40(0) - 330(0)^3\right] = 3.67 \text{ cm}^2/\text{sec}$$

$$\int_{0.1}^{0.2} 40 - 990x^2 \, dx = 40x - 330x^3 \Big|_{0.1}^{0.2} = \left[40(0.2) - 330(0.2)^3\right] - \left[40(0.1) - 330(0.1)^3\right] = 1.69 \text{ cm}^2/\text{sec}$$

A much larger amount of blood flows through the center than near the wall. If the artery gets constricted, then either less blood is able to flow through or the patient's blood pressure increases dramatically, or both. The consequences may be catastrophic.

In summary:
If a function represents an *amount*, the derivative (= slope of the curve) gives us a corresponding *rate of change*.
If a function represents a *rate of change*, the integral (= area under the curve) gives us a corresponding *amount*.