Math 130 Exam 3 Sample 2 Solutions

1. First we take the derivative and clean it up:

$$\begin{aligned}
f(x) &= (x-3)(x-1)^3 \\
f'(x) &= (1)(x-1)^3 + (x-3)3(x-1)^2 \\
f'(x) &= (x-1)^2\Big[(x-1) + 3(x-3)\Big] \\
f'(x) &= (x-1)^2(4x-10)
\end{aligned}$$

   This function is never undefined and equals 0 when $x = 1$ and $x = 5/2$. We test these and the endpoints:

$$\begin{aligned}
f(1) &= (-2)(0)^3 = 0 \\
f(5/2) &= (-1/2)(3/2)^3 = -27/16 \\
f(-2) &= (-5)(-3)^3 = 135 \\
f(3) &= (0)(2)^3 = 0
\end{aligned}$$

   The maximum is then 135 and the minimum is $-27/16$.

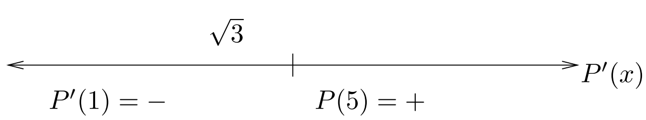
2. The perimeter of the rectangle is $2x + 2y$, however $y = \frac{3}{x}$ and so the perimeter is $P(x) = 2x + 2\left(\frac{3}{x}\right) = 2x + \frac{6}{x}$. This is valid for $x > 0$ (the first quadrant).

   We have $P'(x) = 2 - \frac{6}{x^2}$ which is undefined at $x = 0$ (but this is not in the interval $x > 0$) so we set it equal to 0:

$$\begin{aligned}
2 - \frac{6}{x^2} &= 0 \\
2x^2 - 6 &= 0 \\
x^2 &= 3 \\
x &= \pm\sqrt{3}
\end{aligned}$$

   We throw out $-\sqrt{3}$ because it's not in the interval. Thus we have $x = \sqrt{3}$.

   To see that it's a minimum note that a number line sketch for $P'(x)$ shows us:



   From which we see that $P(x)$ is at its lowest point at $x = \sqrt{3}$.

3. (a) We differentiate implicitly:

$$\begin{aligned}
x^2y^2 - \frac{x}{y} &= 3x \\
\frac{d}{dx}\left[x^2y^2 - \frac{x}{y}\right] &= \frac{d}{dx}[3x] \\
2xy^2 + x^2 2y\frac{dy}{dx} - \frac{1(y) - x(dy/dx)}{y^2} &= 3 \\
2xy^2 + 2x^2y\frac{dy}{dx} - \frac{1}{y} + \frac{x}{y^2}\frac{dy}{dx} &= 3 \\
\frac{dy}{dx}\left[2x^2y + \frac{x}{y^2}\right] &= 3 - 2xy^2 + \frac{1}{y} \\
\frac{dy}{dx} &= \frac{3 - 2xy^2 + 1/y}{2x^2y + x/y^2}
\end{aligned}$$

At $(2, -1)$ this slope is

$$\frac{dy}{dx} = \frac{3 - 2(2)(-1)^2 + 1/(-1)}{2(2)^2(-1) + 2/(-1)^2} = \frac{-2}{-6} = \frac{1}{3}$$

So the line has equation $y - (-1) = \frac{1}{3}(x - 2)$.

(b) We are given $\frac{dA}{dt} = 5$ and we want $\frac{dr}{dt}$. This means we need an equation relating A and r. The area of a circle is $A = \pi r^2$, this does the job. We implicitly differentiate with respect to t and get

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

Now then, when $A = 100$ we have $100 = \pi r^2$ and so $r = \frac{10}{\sqrt{\pi}}$ and so we plug $\frac{dA}{dt}$ and r into our equation:

$$\begin{aligned}
\frac{dA}{dt} &= 2\pi r\frac{dr}{dt} \\
5 &= 2\pi\left(\frac{10}{\sqrt{\pi}}\right)\frac{dr}{dt} \\
5 &= 20\sqrt{\pi}\cdot\frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{1}{4\sqrt{\pi}}
\end{aligned}$$

The units are inches per year.

4. (a) $\int \frac{1}{x} - 2x + 3 \ dx = \ln|x| - x^2 + 3x + C$
   (b) $\int x(x+2)(5x-1) \ dx = \int x^3 + 9x^2 - 2x \ dx = \frac{1}{4}x^4 + 3x^3 - x^2 + C$
   (c) $\int \frac{x^2e^x - 3x + x^5}{x^2} \ dx = \int \frac{x^2e^x}{x^2} - \frac{3x}{x^2} + \frac{x^5}{x^2} \ dx = \int e^x - \frac{3}{x} + x^3 \ dx = e^x - 3\ln|x| + \frac{1}{4}x^4 + C$
   (d) $\int 2^{5x} \ dx = \frac{1}{5\ln 2} \cdot 2^{5x} + C$

5. (a) We let

$$\begin{aligned} u &= 1 + 4\cos x \\ \frac{du}{dx} &= -4\sin x \\ du &= -4\sin x \ dx \\ -\frac{1}{4} \ du &= \sin x \ dx \end{aligned}$$

So then

$$\int \sin x e^{(1+4\cos x)} \ dx = \int e^u \left(-\frac{1}{4} \ du\right) = \int -\frac{1}{4}e^u \ du = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{(1+4\cos x)} + C$$

   (b) We let

$$\begin{aligned} u &= 4x - 8 \\ \frac{du}{dx} &= 4 \\ du &= 4 \ dx \\ \frac{1}{4} \ du &= dx \end{aligned}$$

Note we also need $x = \frac{u+8}{4}$ here. So then

$$\begin{aligned} \int \frac{x^2}{4x-8} \ dx &= \int \frac{((u+8)/4)^2}{u} \frac{1}{4} \ du \\ &= \int \frac{1}{64} \cdot \frac{u^2 + 16u + 64}{u} \ du \\ &= \int \frac{1}{64} \left[u + 16 + \frac{64}{u}\right] \ du \\ &= \frac{1}{64} \left[\frac{1}{2}u^2 + 16u + 64\ln|u|\right] + C \\ &= \frac{1}{64} \left[\frac{1}{2}(4x-8)^2 + 16(4x-8) + 64\ln|4x-8|\right] + C \end{aligned}$$

(c) Since the rate $D'(t)$ is given we have $D'(t) = 4t + 5$. We find the antiderivative

$$D(t) = \int D'(t)\ dt = \int 4t + 5\ dt = 2t^2 + 5t + C$$

Thus $D(t) = 2t^2 + 5t + C$. Now then we know $D(12) = 100$ and so

$$\begin{aligned} 2(12)^2 + 5(12) + C &= 100 \\ 288 + 60 + C &= 100 \\ C &= -248 \end{aligned}$$

Thus

$$D(t) = 2t^2 + 5t - 248$$

Then after two years there are $D(24) = 2(24)^2 + 5(24) - 248$ people with the disease.