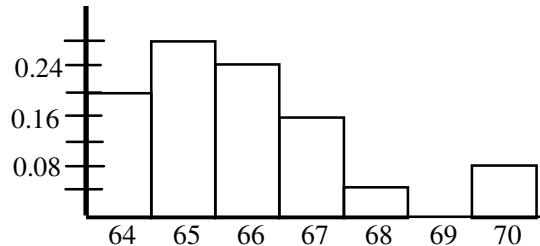# Calculus 131, section 13.1 Continuous Random Variables
notes by Tim Pilachowski

Suppose we measure the heights of 25 people to the nearest inch and get the following results:

| height (in.)	64	65	66	67	68	69	70
frequency	5	7	6	4	1	0	2
probability	0.2	0.28	0.24	0.16	0.04	0	0.08

The probability distribution graph would look like this:

Recall that the sum of the areas of the rectangles, which is the sum of the probabilities, equals 1.
(probability as area: area of each bar = height times width = probability times 1 = percentage of people having
that height = relative frequency of that height = probability of a person having that height)

If we were to measure to the nearest half-inch, or tenth of an inch, or hundredth, or thousandth, etc., etc., etc.,
we'd get ever-more-narrow rectangles, and would get something more like a curve:
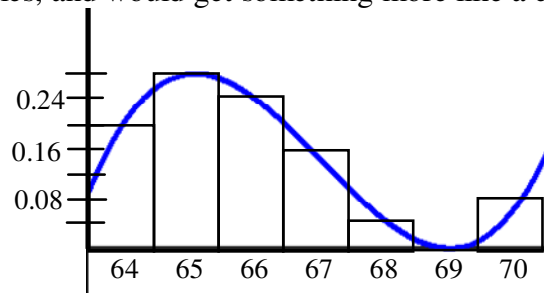
The area under the curve for a given interval would be the probability of people having heights within that
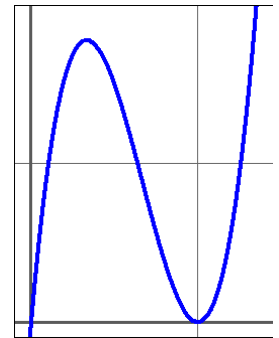interval.

And so we come to a definition—a *probability density function f(x)* has two necessary characteristics:
1. $f(x) \geq 0$ for all values of x in its domain, the interval $[a, b]$ [since all probabilities, and therefore areas
under the curve, are zero or positive]

2. $\int_a^b f(x)\,dx = 1$ (since the sum of all probabilities = 1 = area under the curve over the entire domain)

Example A: Verify that $f(x) = 12x(1 - x)^2$ is a probability density function on [0, 1]. *answer*: Yes, it is.

Example B: Find the value of the constant k that makes $f(x) = kx(1 - x)$ a probability density function on the interval $0 \leq x \leq 1$. *answer*: 6

Example A revisited: A continuous random variable X has the probability density function $f(x) = 12x(1 - x)^2$, $0 \leq x \leq 1$. Find $P\left(0 \leq x \leq \frac{1}{2}\right)$. *answer*: $\frac{11}{16} = 0.6875$ *Note both versions are exact.*

Example A once more: A continuous random variable X has the probability density function $f(x) = 12x(1 - x)^2$, $0 \leq x \leq 1$. Find $P\left(\frac{1}{2} \leq x \leq 1\right)$. *answer*: $\frac{5}{16} = 0.3125$
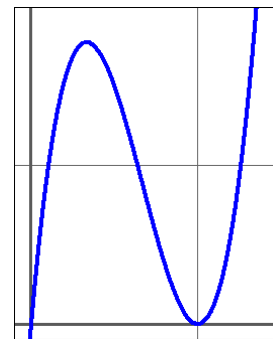
Note that we are finding probabilities for intervals as opposed to specific values. Indeed, if we tried to find $P\left(x = \frac{1}{2}\right)$, the calculus would give us $\int_{1/2}^{1/2} 12x\left(1 - x^2\right) dx = 0$. This isn't to say that $P(x)$ could never equal zero, but rather the probability for that one specific value is so small that it is negligible. Parallels in life include:
- A meteorologist will predict rain for the afternoon, not rain for 2:07 pm.
- The bullseye on a dartboard is a space, not an infinitesimally small point.
- A person growing from 68 to 70 inches will at some time be exactly 69 inches, but we have no way to measure with that much accuracy to specify exactly when it happens.

Example C: Given a random variable X with probability density function $f(x) = \frac{1}{9}x^2$ on $0 \le x \le 3$, find $P(1 \le x \le 2)$.   *answer*: $\frac{7}{27}$

Example D. For a particular machine, its useful lifetime (random variable X) is modeled by $f(t) = 0.1e^{-0.1t}$, $0 \le t \le \infty$. Verify that f is a probability density function, then find the probability that the machine will last a) less than 2 years, b) more than 2 but less than 4 years, and c) more than 4 years.

answers: a) $e^{-0.2} + 1 \cong 0.181$, b) $-e^{-0.4} + e^{-0.2} \cong 0.148$, c) $e^{-0.4} \cong 0.670$

side note: The three probabilities add to 1, as they should. (We must add the exact values, because the approximate answers have a rounding error.)