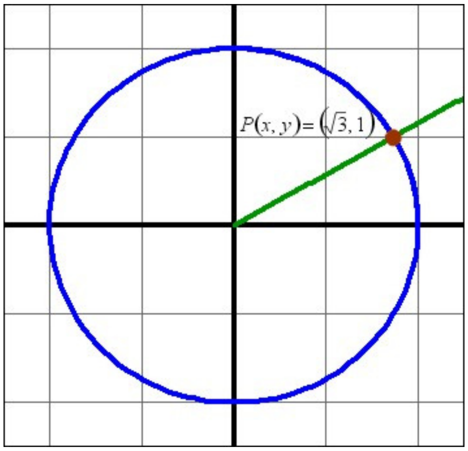
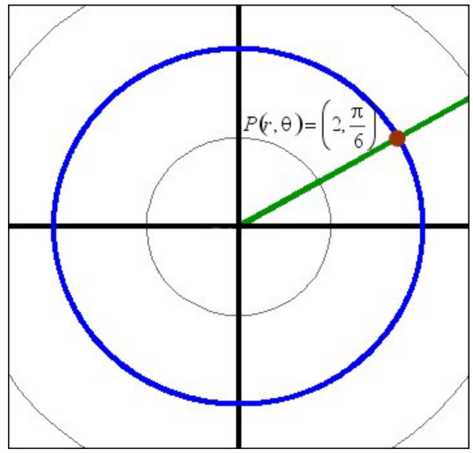
# Calculus 141, section 10.1 Polar Coordinates Introduction

notes by Tim Pilachowski

Before we begin, recall the earlier material on parametric equations, trigonometric functions, and polar coordinates. We'll need it.

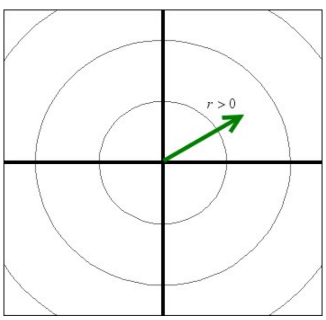
Any point $(x, y)$ in the Cartesian plane can also be located by using its distance from the origin $(r)$ and the angle of inclination used to get there $(\theta)$. Consider, for example, the point which has rectangular coordinates $P(x,\ y) = \left(\sqrt{3}, 1\right)$ and polar coordinates $P(r, \theta) = \left(2, \dfrac{\pi}{6}\right)$.

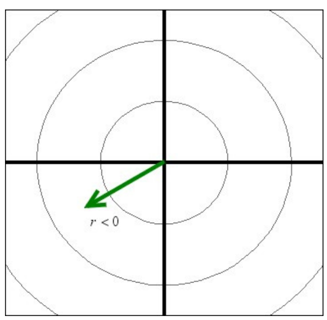


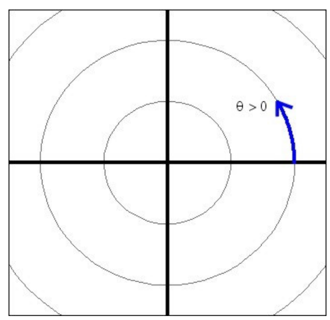



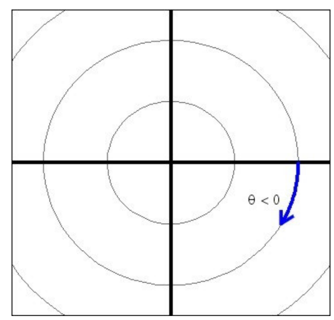

If you are familiar with vectors you will recognize the polar coordinates as expressing magnitude and direction. (One application which uses such vectors is the designation of phasors in measuring voltages and currents in alternating current electricity.)

Polar coordinates are a *directed* distance and a *directed* angle. The magnitude, $r$, is positive to the right of the origin, and negative to the left of the origin.



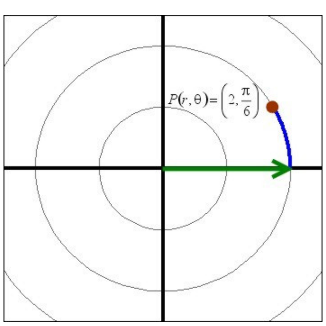



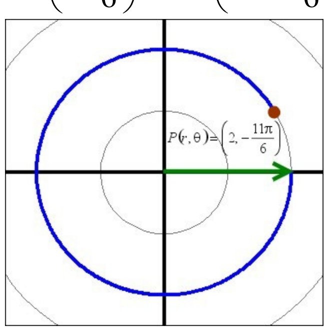



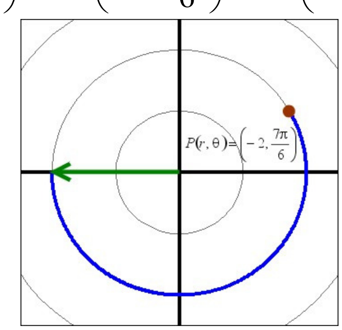

As in trigonometry, $\theta$ is positive when measured counterclockwise and negative when measured clockwise.

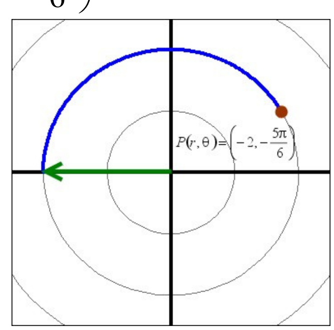
Thus, any $(x, y)$ point will have numerous designations in polar coordinates:

$$P(r, \theta) \quad = \quad \left(2, \frac{\pi}{6}\right) \quad = \quad \left(2, -\frac{11\pi}{6}\right) \quad = \quad \left(-2, \frac{7\pi}{6}\right) \quad = \quad \left(-2, -\frac{5\pi}{6}\right).$$



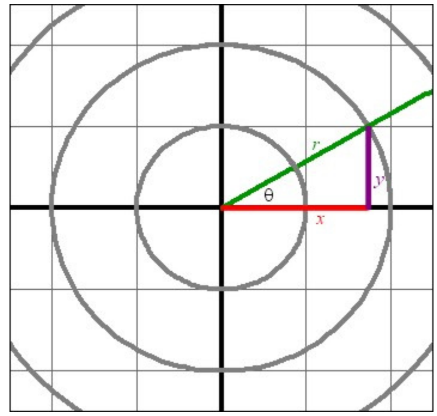




If we wanted a unique designation we could restrict ourselves to $r > 0$ and $0 \le \theta < 2\pi$.

Placing both rectangular and polar coordinates, $P(x, y) = P(r, \theta)$, on the same Cartesian plane helps to illustrate the relationship between these two ways of identifying a point. Using the triangle ratios applied to trigonometric functions,

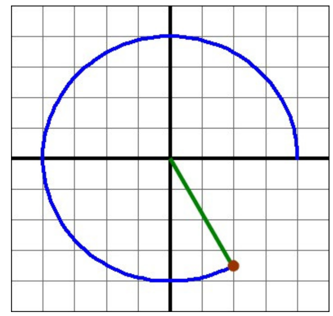



$\cos\theta = \dfrac{x}{r} \Rightarrow x = r\cos\theta$ $\qquad\qquad$ $\sin\theta = \dfrac{y}{r} \Rightarrow y = r\sin\theta$

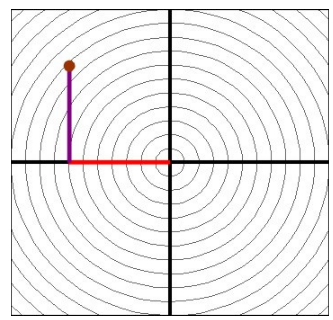
$\tan\theta = \dfrac{y}{x}$ $\qquad\qquad$ and $r^2 = x^2 + y^2 \Rightarrow \quad r = \pm\sqrt{x^2 + y^2}$ .

A circle of radius $a$ will have the equation $r = a$ and a line through the origin with angle of inclination α will have the equation $\theta = \alpha$.

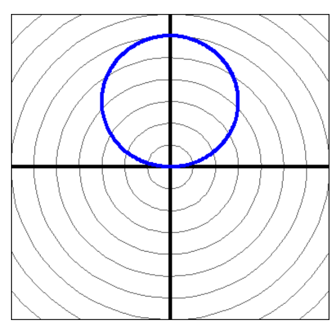
Example A: Convert polar coordinates $\left(4, \dfrac{5\pi}{3}\right)$ to rectangular coordinates.



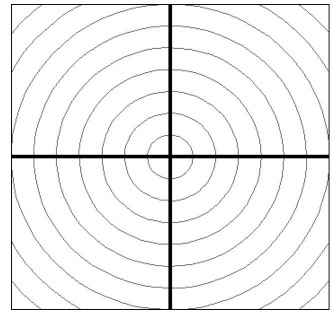
Example B: Convert rectangular coordinates (–7, 7) to polar coordinates. Note there will be multiple answers.



Example C: Find a polar equation for the circle $x^2 + (y-3)^2 = 9$ [circle with radius 3, centered at (0, 3)].



Example D: Find a rectangular coordinate equation for $r = \dfrac{4}{2\cos\theta - \sin\theta}$.

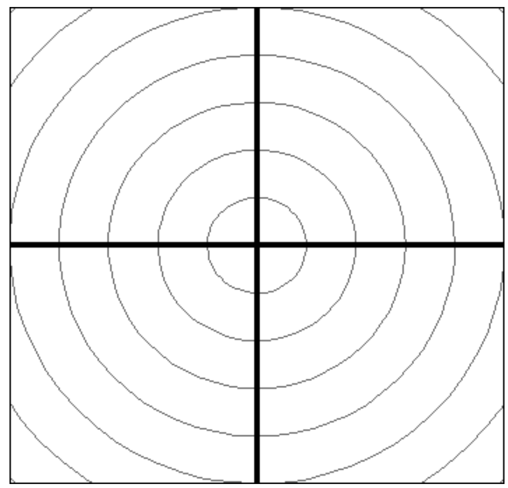


In general, rectangular coordinates are better-suited to lines than rectangular coordinates.

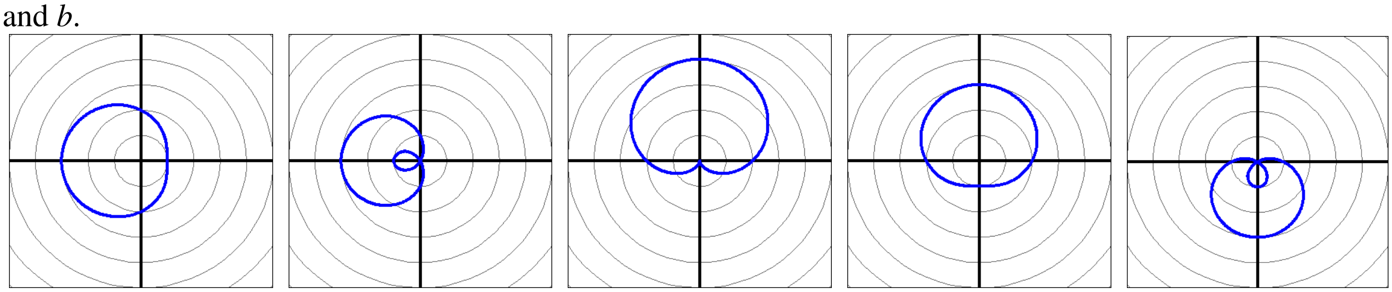
Polar coordinates work better for other types of equations, particularly those which would not be a function if expressed in rectangular coordinates. One item to note: The convention in rectangular coordinates is to interpret the point $(x, y)$ as $y = f(x)$, i.e. the second coordinate is a function of the first. However, when using polar coordinates, the convention is to interpret the point $(r, \theta)$ as $r = f(\theta)$, i.e. the first coordinate is a function of the second. Under this convention, a positive value for $r$ means travel from the origin toward the angle $\theta$. A negative value for $r$ means travel from the origin in a direction opposite the angle $\theta$.

Example E: Graph the polar equation $r = 2 - 2\cos\theta$.

| $\theta$ | $0$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | | | | | |



Graphs of functions of the form $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ are called limaçons, with the special case for which $a = b$ (like Example E) called cardioids. The graphs below illustrate the effects of changing the constants $a$ and $b$.

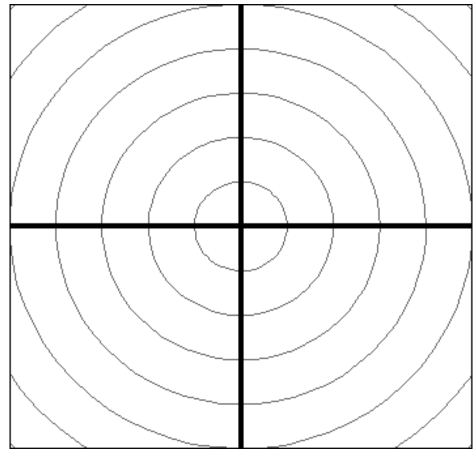


$r = 2 - \cos\theta$ $\quad$ $r = 1 - 2\cos\theta$ $\quad$ $2 + 2\sin\theta$ $\quad$ $r = 2 + \sin\theta$ $\quad$ $r = 1 - 2\sin\theta$

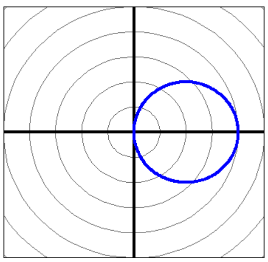
Note that the graphs $r = 1 - 2\cos\theta$ and $r = 1 - 2\sin\theta$ have "inner loops", corresponding to negative values of $r$. Also note that the three graphs involving $\sin\theta$ are symmetric with respect to the $y$-axis, i.e. the line $\theta = \dfrac{\pi}{2}$.

Example F: Graph the equation $r = 4\cos(2\theta)$.

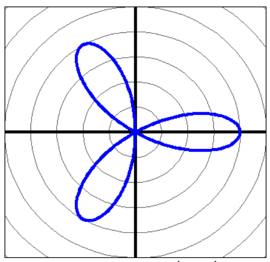
| $\theta$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $3\pi/4$ | $5\pi/6$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | | | | | | | | | |



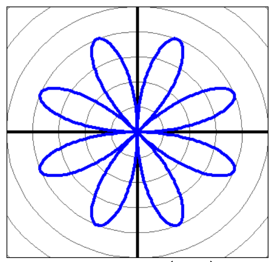
(Note the symmetry created by the coefficient 2! The negative values of cos($2\theta$), found when $2\theta$ is in QII, provide negative values for *r*. These negative values place the points across the origin opposite the angle $\theta$. The shape of this graph is called a four-leaved rose. Here are graphs for polar functions with other coefficients of $\theta$.



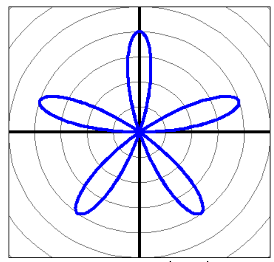
$r = 4\cos\theta$



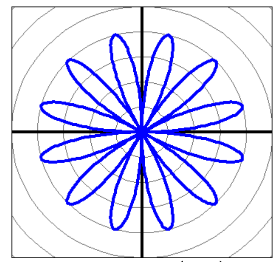
$r = 4\cos(3\theta)$



$r = 4\sin(4\theta)$



$r = 4\sin(5\theta)$

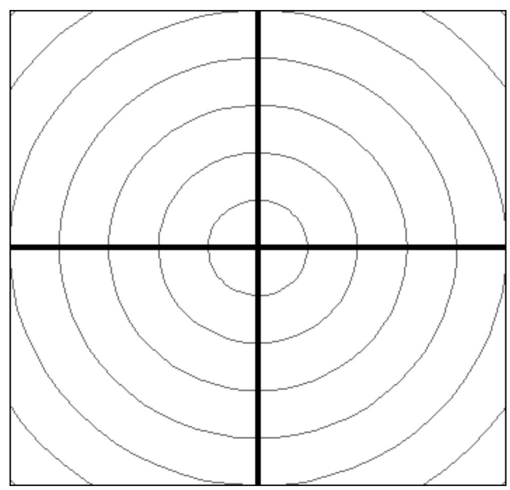


$r = 4\sin(6\theta)$

You can use a graphing calculator (see instructions below) to investigate coefficients that are fractions and irrational numbers.

Example G: Graph the lemniscate (Latin for "ribbon") $r^2 = 16\cos\theta$.

| $\theta$ | 0 | $\pi/3$ | $\pi/2$ | $\pi/2 < \theta < 3\pi/2$ | $3\pi/2$ | $5\pi/3$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $r$ | | | | | | | |



The text summarizes symmetry tests in Table 10.1 and also provides a "dictionary" of polar graphs at the end of the chapter.

You can set your graphing calculator to do polar graphs in one of two ways. Many graphing calculators use the variable *t* in place of $\theta$.

1) As polar equations in *r* and *t*: On the MODE screen choose **Pol**. Your equation(s) will need to be entered in the form "$r = f(t)$". This lemniscate would be entered as $r_1 = \sqrt{16\cos t}$ and $r_2 = -\sqrt{16\cos t}$.

2) As parametric equations for *x* and *y* in terms of *t*: On the MODE screen choose **Param**. Your equations will need to be entered in the forms "$x = f(t) * \cos t$" and "$y = f(t) * \sin t$". This lemniscate would be entered as $x_1 = \sqrt{16\cos t} * \cos t$, $x_2 = -\sqrt{16\cos t} * \cos t$, $y_1 = \sqrt{16\cos t} * \sin t$, and $y_2 = -\sqrt{16\cos t} * \sin t$.