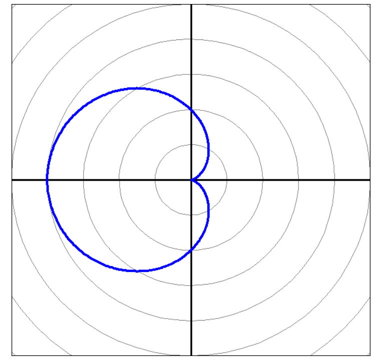
Calculus 141, section 10.2 Polar Equation Length and Area

notes by Tim Pilachowski, Fall 2004

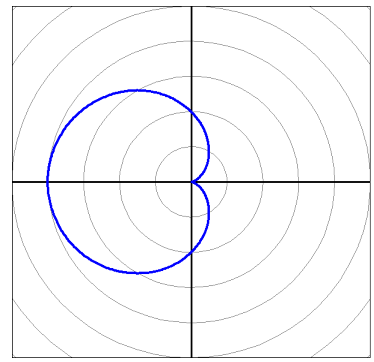


Back in Math 140 (chapter 5) we found the area of a region in rectangular coordinates by partitioning the region into rectangles. With polar coordinates we'll need to think radially, and partition the region into pie-shaped sectors. The area of each sector is

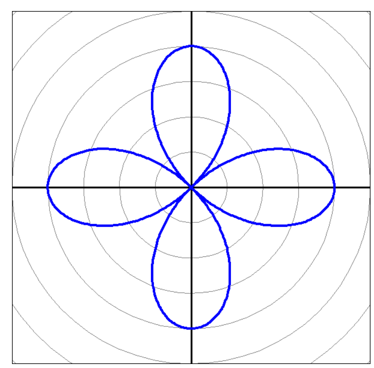
$$\text{(fraction of circle) times (area of circle)} = \frac{\Delta\theta}{2\pi} * \pi r^2 = \frac{r^2}{2} * \Delta\theta .$$

The sum of these areas is a Riemann sum, thus we get $A = \int_{\alpha}^{\beta} \frac{[f(\theta)]^2}{2}\, d\theta$.

10.1 Example E again: Find the area of the region enclosed by $r = 2 - 2\cos\theta$.
Answer: 6π



10.1 Example F revisited: Find the area of the region enclosed by $r = 4\cos(2\theta)$. *Answer*: 8π

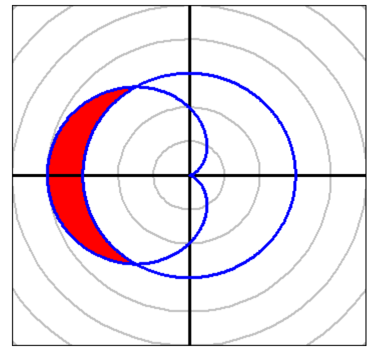


Related topic: The area of the region between two polar curves is given by $A=\int_{\alpha}^{\beta}\frac{1}{2}\left\{[f(\theta)]^2-[g(\theta)]^2\right\}d\theta$.

In rectangular coordinates, we said "above" – "below". In polar coordinates, we'll say "outside" – "inside".

Example B: Find the area of the region that lies inside the cardioid $r=2-2\cos\theta$ and outside the circle $r=3$.

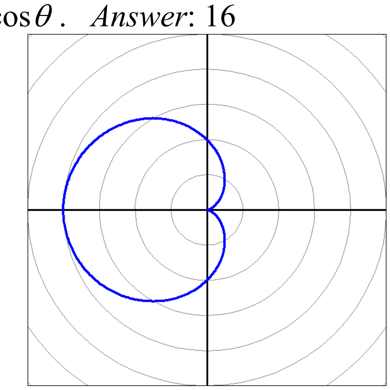
Answer: $-\pi+\frac{9\sqrt{3}}{2}$



Since a polar equation $r=f(\theta)$ can be written parametrically as $x=f(\theta)*\cos\theta$ and $y=f(\theta)*\cos\theta$, we can use work done previously in section 6.8 to determine the length of a polar curve over the interval $[\alpha, \beta]$.

$L=\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2+[f'(\theta)]^2}\,d\theta=\int_{\alpha}^{\beta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$. (See the text for all the steps.)

10.1 Example E revisited: Find the length of one traversal of the curve $r=2-2\cos\theta$. *Answer*: 16



Example A: Find the length of the curve $r = e^{\theta/4}$ for $\theta = 0$ to 2π. *Answer*: $r = \sqrt{17}\left(e^{\pi/2} - 1\right)$