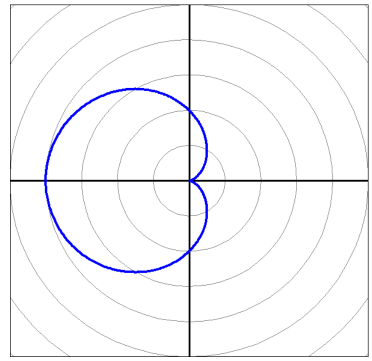
Calculus 141, section 10.2 Polar Equation Length and Area

notes by Tim Pilachowski, Fall 2004

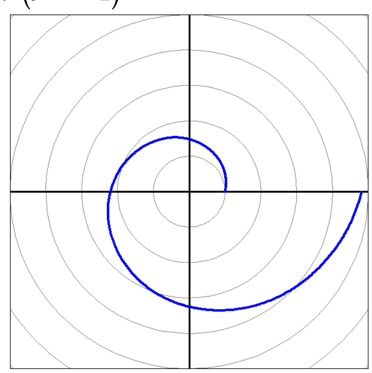
Since a polar equation $r = f(\theta)$ can be written parametrically as $x = f(\theta) * \cos\theta$ and $y = f(\theta) * \cos\theta$, we can use work done previously in section 6.8 to determine the length of a polar curve over the interval $[\alpha, \beta]$.

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$ (See the text for all the steps.)

10.1 Example E revisited: Find the length of one traversal of the curve $r = 2 - 2\cos\theta$. *Answer*: 16



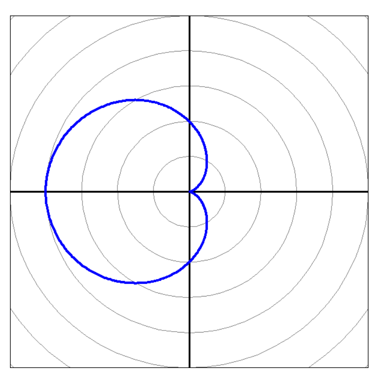
Example A: Find the length of the curve $r = e^{\theta/4}$ for $\theta = 0$ to 2π. *Answer*: $r = \sqrt{17}\left(e^{\pi/2} - 1\right)$



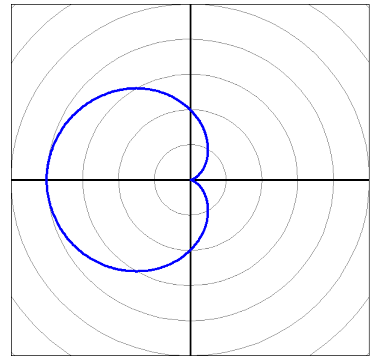
Back in Math 140 (chapter 5) we found the area of a region in rectangular coordinates by partitioning the region into rectangles. With polar coordinates we'll need to think radially, and partition the region into pie-shaped sectors. The area of each sector is

(fraction of circle) times (area of circle) = $\dfrac{\Delta\theta}{2\pi} * \pi r^2 = \dfrac{r^2}{2} * \Delta\theta$.

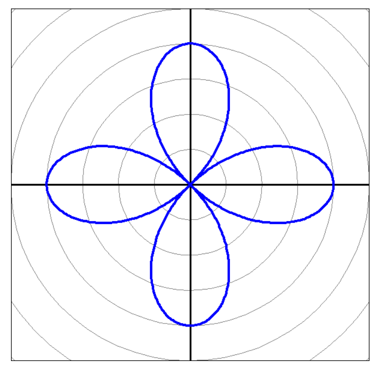
The sum of these areas is a Riemann sum, thus we get $A = \int_{\alpha}^{\beta} \dfrac{[f(\theta)]^2}{2}\, d\theta$.



10.1 Example E again: Find the area of the region enclosed by $r = 2 - 2\cos\theta$.



10.1 Example F revisited: Find the area of the region enclosed by $r = 4\cos(2\theta)$. *Answer*: 6π



Related topic: The area of the region between two polar curves is given by $A = \int_\alpha^\beta \frac{1}{2}\left\{[f(\theta)]^2 - [g(\theta)]^2\right\} d\theta$.

In rectangular coordinates, we said "above" – "below". In polar coordinates, we'll say "outside" – "inside".

Example B: Find the area of the region that lies inside the cardioid $r = 2 - 2\cos\theta$ and outside the circle $r = 3$.

Answer: $-\pi + \frac{9\sqrt{3}}{2}$