Calculus 141, section Complex.1 Intro and Basics

notes by Tim Pilachowski

Be sure to read, print and/or download the 141 complex number notes (C#N). [You can also link to them either from our Math 141 web page or from the Math Department's course information page.] There's also a 141 complex number summary. The lecture below contains material from sections 1 – 4, 6, 7, and 11 – 13 of the C#N. (The related exercises in C#N are numbers 1 – 4, 7, 8, 10 and 12.)

Any Real number can be signified by a physical representation. Integers greater than zero can be represented by the counting of objects. Positive and negative can be represented by upward/downward or forward/backward movement. Rational numbers come from whole objects cut into equal-sized pieces. A square root is the hypotenuse of an appropriate right triangle. π is the circumference of a circle divided by its diameter. e is modeled by, and is the basis of the mathematical model for, exponential growth and decay.

On the other hand, i exists only in the human imagination. You probably first encountered imaginary roots when looking for x-intercepts of a function such as $f(x)=x^2+x+1$. The quadratic formula provides roots of $-\frac{1}{2}\pm\frac{\sqrt{-3}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$, but the graph on the Cartesian plane has no x-intercepts.

However, even though imaginary numbers are imaginary, they are still quite useful. For example, Geographic Information Systems (GIS) ,designed to capture, store, manipulate, analyze, manage, and present spatial or geographic data, rely on conformal (or angle-preserving) mapping, which uses an algorithm in complex numbers. (source: Dr. Gary Felton, UMCP Department of Environmental Science and Technology). For an electrician, alternating current amplitude and phase shift are represented by complex number forms (source: a course I taught many years ago for electrician certification).

To start at the beginning, we define the imaginary number $i=\sqrt{-1}$. Thus $i^2=-1$, $i^3=-\sqrt{-1}=-i$, $i^4=-i^2=1$, and $i^5=i$. A complex number, of the form $z = a + bi$, has a "real" number part, a, and an "imaginary" part, bi, where b is a real number coefficient and i is defined as above. Note that the set of Real numbers $\mathbb{R}$ is a proper subset of the set of Complex numbers $\mathbb{C}$, since any Real number a can be written in the complex form $a + 0i$.

The arithmetic of $\mathbb{C}$ follows all the same rules and has all the same properties as the arithmetic of $\mathbb{R}$.

Examples A: Find $(2-3i)+(4+5i)$ and $(2-3i)-(4+5i)$. *Answers*: $6 + 2i$; $-2 - 8i$

Example B: Find $(2-3i)*(4+5i)$. *Answer*: $23 - 2i$

Division in $\mathbb{C}$ is a bit more complicated. We can define it as multiplication by a multiplicative inverse (reciprocal), but first we must investigate what a Complex reciprocal would look like. Before we can do that, we need to define the **conjugate** of a complex number. Given $z = a + bi$, its conjugate is defined as $\bar{z}=a-bi$. Note that $z*\bar{z}$ gives us the difference of two squares:

$$z*\bar{z}=(a+bi)(a-bi)=a^2-(bi)^2=a^2-(-b^2)=a^2+b^2.$$

Example C: Find the reciprocal of the complex number 4 + 5i and express it in the form a + bi.
Answer: $\frac{4}{41} - \frac{5}{41}i$

Example D: Find $\frac{2-3i}{4+5i}$ and express the answer in the form a + bi. *Answer*: $-\frac{7}{41} - \frac{22}{41}i$

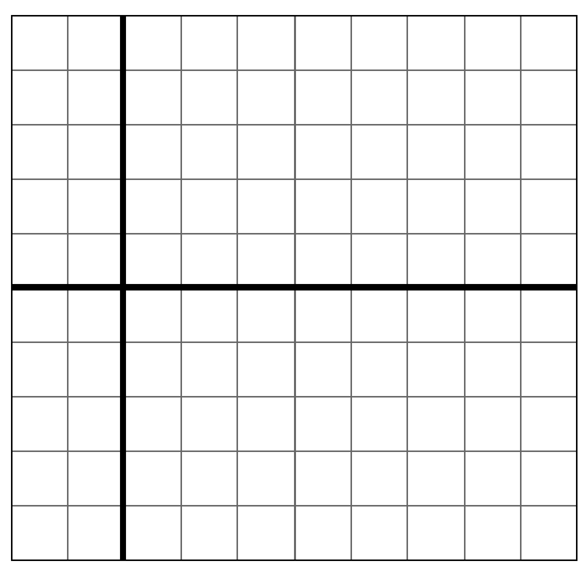
Exercise 3 in the C#N is further investigations of multiplicative inverse and division of complex numbers.

Even though i does not represent any entity in the physical world, it is possible to design a physical representation of complex numbers. We can diagram the complex number $z = a + bi$ in the complex plane as vectors in either rectangular coordinates (a on the horizontal axis and b on the vertical) or as polar coordinates (r = distance/magnitude, θ = angle/direction).

In the rectangular coordinate scenario, the absolute value of z is defined as the distance from the origin to z: $|z| = \sqrt{a^2 + b^2}$. [Side note: In other texts or classes, you may see this referred to as the **modulus** or **norm** of z.]

In the polar coordinate scheme, the distance from the origin to $z = r$. Note also that since $a = r\cos\theta$ and $b = r\sin\theta$, we can write that $z = a + bi = r\cos\theta + i(r\sin\theta)$.

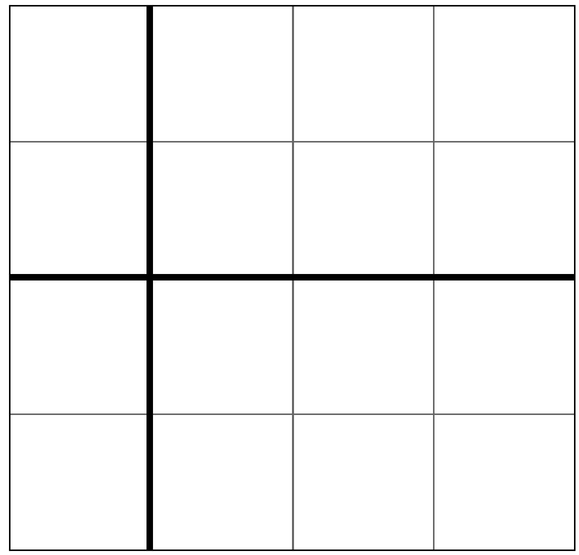
Example A revisited: Diagram (2 – 3i) + (4 + 5i).



A visual explanation of the multiplication of complex numbers is more easily done in polar coordinates than in rectangular, but needs a little more background. Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$ allows us to write $z = a + ib = r\cos\theta + i\,r\sin\theta = re^{i\theta}$. [More on this in Lecture Complex.2.]

The product of two complex numbers would give us $(z_1)(z_2) = \left(r_1 e^{i\alpha}\right)\left(r_2 e^{i\beta}\right) = (r_1\, r_2)e^{i\,(\alpha+\beta)} = (r_1\, r_2)\cos(\alpha+\beta) + i(r_1\, r_2)\sin(\alpha+\beta)$. The product's distance from the origin is the product of the two distances: $(r_1\, r_2) = |\, z_1\, z_2\,| = |\, z_1\,| * |\, z_2\,|$. The product's angle of rotation is the sum of the two angles.

Example E: Verify the product $z_1\, z_2 = (1+i)\left(\sqrt{3} - i\right) = 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$.



See section 13 of the complex number notes for a proof of the sum-of-angles identities.

Example F: Given $z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, find z^{10}. *Answer*: $-i$