Calculus 141, section Complex.2 Extended Basics

notes by Tim Pilachowski

Be sure to read, print and/or download the 141 complex number notes (C#N). [You can also link to them either from our Math 141 web page or from the Math Department's course information page.] There's also a 141 complex number summary. The lecture below contains material from sections 5, 8, 9, and 10 of the C#N. (The related exercises in C#N are numbers 5, 6, 9, 11 and 13 – 16.)

Just like for Real numbers, geometric series with Complex number terms will have a radius of convergence $0 \le R < \infty$ in the Complex plane where R is the radius of a circle centered at the origin. The geometric series $\sum_{n=0}^{\infty} z^n$ will converge to $\frac{1}{1-z}$ when $|z| < 1$.

Example A: Find a and b such that $\sum_{n=0}^{\infty}\left(\frac{2}{5}+\frac{1}{5}i\right)^n = a+bi$. *Answers*: $\frac{3}{2}$; $\frac{1}{2}$

Example B: Prove Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$.

Considered as a function in the complex plane, $f(\theta)=e^{i\theta}$ is a circle of radius 1 centered at the origin (the unit circle). In particular, note that $f(\pi)=e^{i\pi}=\cos\pi+i\sin\pi=-1+0=-1$, which leads to $e^{i\pi}+1=0$, a quite elegant result.

Derivatives of functions involving Complex numbers have the same properties and follow the same conventions as for Real numbers, thus $\frac{d}{d\theta}\left(e^{i\theta}\right)=ie^{i\theta}=i(\cos\theta=i\sin\theta)=i\cos\theta-\sin\theta=-\sin\theta+i\cos\theta$.

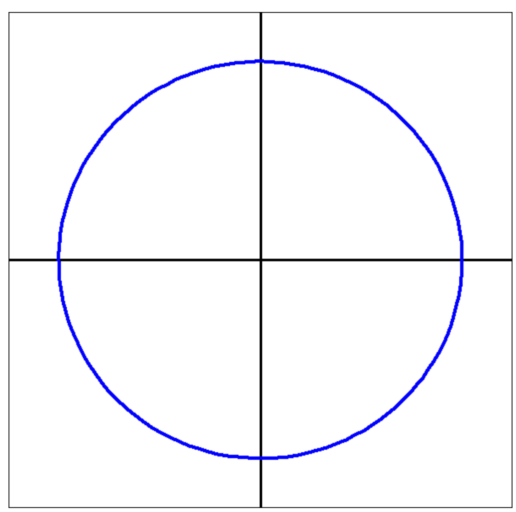
Euler's formula led to $z=r\cos\theta+ir\sin\theta=re^{i\theta}$. The product of two complex numbers was then $(z_1)(z_2)=\left(r_1e^{i\alpha}\right)\left(r_2e^{i\beta}\right)=(r_1\,r_2)e^{i\,(\alpha+\beta)}$. Now consider the case for which $z=\cos\theta+i\sin\theta$ and we want to calculate $z^n=(\cos\theta+i\sin\theta)^n$. The observation about products above leads us to DeMoivre's Theorem: $z^n=(\cos\theta+i\sin\theta)^n=\left(e^{i\theta}\right)^n=e^{i(n\theta)}=\cos(n\theta)+i\sin(n\theta)$. One result of DeMoivre's Theorem is that we now have a means of deriving formulae for $\cos(n\theta)$ and $\sin(n\theta)$.

Example C: Verify the double-angle formulae for cosine and sine.

A similar process of computation and comparison would provide us formulae for $\cos(n\theta)$ and $\sin(n\theta)$ as polynomials of degree n in cos θ and sin θ. (See exercise #6 in the C#N.)

Given a complex number $z=re^{i\theta}\neq 0$ and an integer $n > 0$, there are precisely n distinct n^{th} roots of z. Let $w=\rho e^{i\alpha}$ represent an n^{th} root of z. Then $z=w^n \Rightarrow re^{i\theta}=\rho^n e^{in\alpha}$. Thus $\rho^n=r \Rightarrow \rho=\sqrt[n]{r}$, that is, ρ is the real, positive n^{th} root of r. Also, $n\alpha=\theta+2k\pi \Rightarrow \alpha=\frac{\theta}{n}+k\frac{2\pi}{n}$.

Note that the roots w_k lie on a circle of radius ρ centered at the origin. If we take values for k = 0, 1, 2, … , n – 1, we will travel around that circle to n uniformly-spaced points, each of which is separated from its neighbors by an angle of $\frac{2\pi}{n}$. Our method: Find w_0 (the smallest answer ≥ 0), then add $\frac{2\pi}{n}$ to the angle until we have all n roots.



Example D: Find the 3 cube roots of -27.

Example E: Find a) the 4 fourth roots of 81, and b) the 4 fourth roots of $81i$.

Finding the nth roots of 1 is investigated in exercise #11 in the C#N. These "roots of unity" can be found either by solving an equation as in Example D or E above, or by beginning on the unit circle at $z = 1 + 0i$ and moving around the circle in successive angles $\frac{2\pi}{n}$.

Section 5 of the C#N refers to the Fundamental Theorem of Algebra (Every nonconstant polynomial with coefficients in $\boldsymbol{C}$ (or $\boldsymbol{R}$) has roots in $\boldsymbol{C}$.), and the Factorization Theorem:
A polynomial with complex coefficients of degree at least 1, $p(z) = c_n z^n + c_{n-1} z^{n-1} + c_{n-2} z^{n-2} + \ldots + c_1 z + c_0$ can be factored as a product of linear terms, $p(z) = c_n (z - z_1)(z - z_2)\ldots(z - z_n)$, some of which may have a multiplicity greater than one.

Example F: Find the linear factors of $z^3 + 27$.

Example G: Find the linear factors of $z^4 + 2z^2 - 3$.