

Calculus 141, section 6.0 (quick review) & 6.1 Volume

notes by Tim Pilachowski

Notes for each lecture will be posted on my Math Department website, www2.math.umd.edu/~tjp, prior to the lecture itself. You should print out and/or download each of these and bring it with you to class. In this way you can put your attention on listening and thinking, and only need to write all those little “extras” that will come up during my presentation. **Need I tell you that these notes will be an outline only, and that they cannot replace your presence in the lecture?**

Be sure to attend the discussions on a regular basis, too. You’ll find them to be valuable in cementing the topics covered in the lecture. You’ll get the most out of the discussion if you do the assigned homework *before* the discussion, and *participate* in all the discussion activities.

To help you get up to speed for Math 141, we’re going to spend this first class going over some things I assume you already know, but about which you may need a little reminder. Go to the Math Dept. Testbank (follow the Math 140 link from www2.math.umd.edu/~tjp and download the Math 140 Final Exam from last semester and try to answer all the questions.

The following statements are mathematically equivalent:

- a) Find the slope of the line tangent to the graph of f at a point (x, y) .
- b) Find $\lim_{a \rightarrow b} \frac{f(b) - f(a)}{b - a} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$. c) Find the first derivative of $f(x)$.
- d) Find $f'(x)$. e) Find $\frac{dy}{dx}$.

Recall, however, that the first derivative is itself a function, which has its own domain and graph. Since it is a function, it has its own derivative. Given a function f , we can calculate the first derivative f' or $\frac{dy}{dx}$. We can then calculate the derivative of f' , also called the second derivative of f , symbolically f'' or $\frac{d^2y}{dx^2}$.

Important note: Just like $\frac{dy}{dx}$ is *not* a fraction, but is a notation for the first derivative, $\frac{d^2y}{dx^2}$ is also not a fraction but a notation. *There is no multiplication involved!* Rather, you need to interpret it this way:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) \text{ which means “the derivative of } \frac{dy}{dx} \text{”, the derivative of a derivative.}$$

Example A: Find the following limits: a) $\lim_{x \rightarrow 0^+} \sqrt{e^x - 1}$ b) $\lim_{x \rightarrow 0^-} \sqrt{e^x - 1}$. *Answers:* 0; does not exist

Example B: Given $f(x) = (5x^4 - 1)^2$ find y when $x = -1$, $f'(-1)$ and $\left. \frac{d^2y}{dx^2} \right|_{x=-1}$. *Answers: 16; -160; 1280*

Example C: Given $y = (2x + 1)(\sqrt{x} - 1)$ find $\frac{dy}{dx}$.

Example D: Find $\int (3x^{-6} - 2e^{5x} + 4x^{-1} - 7) dx$. *Answer: $-\frac{3}{5}x^{-5} - \frac{2}{5}e^{5x} + 4\ln|x| - 7x + C$*

Example E: Determine $\int \tan^2 t dt$. *Answer: $\tan t - t + C$*

Example F: Find $\int_1^{e^2} \frac{\ln x}{2x} dx$. *answer: 1*

Example G: $\int \frac{x}{e^{x^2}} dx$.

One might be tempted to approach this integral as a case of $\int \frac{1}{u} du$, since e^{x^2} is in the denominator. It will be

more helpful to first rewrite $\int \frac{x}{e^{x^2}} dx = \int x e^{-x^2} dx$, and let $u = -x^2$, $du = -2x dx \Rightarrow -\frac{1}{2} du = x dx$.

$$\int \frac{x}{e^{x^2}} dx = \int e^{-x^2} (x) dx = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C.$$

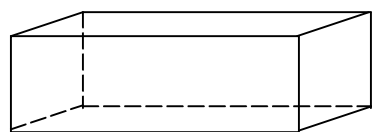
Example H: $\int \sin^2(4x) \cos(4x) dx$.

In this case, one might think that since sine and cosine are essentially derivatives of each other, the choice for u is arbitrary. Actually, $u = \cos(4x)$ would not work well, because its derivative involves $\sin(4x)$ and *not* $\sin^2(4x)$. The only workable choice is to let $u = \sin(4x)$ and thus $du = 4\cos(4x) dx$. Then

$$\int \sin^2(4x) \cos(4x) dx = \frac{1}{4} \int u^2 du = \frac{1}{4} * \frac{1}{3} u^3 + C = \frac{1}{12} \sin^3(4x) + C.$$

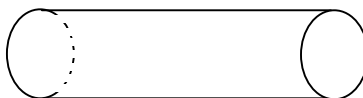
Here ends the review and begins **SECTION 6.1**.

From your knowledge of basic geometry, you are probably already familiar with the formula for volume of a rectangular prism: length times width times height. If we think of that prism as lying on its side, the formula looks more like (volume) = (area of cross-section) times (length).



length

cross-section
is rectangular
 $A = \text{width} * \text{height}$



length

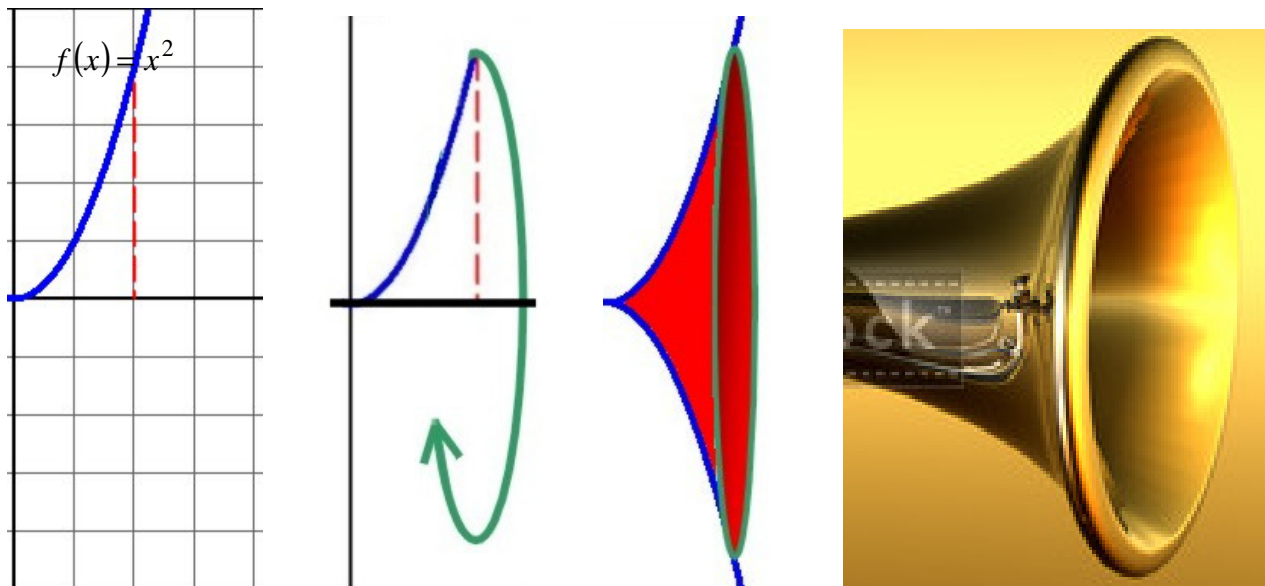
cross-section
is circular
 $A = \pi * r^2$

The volume of a cylinder is the same: (area of cross-section) times (length).

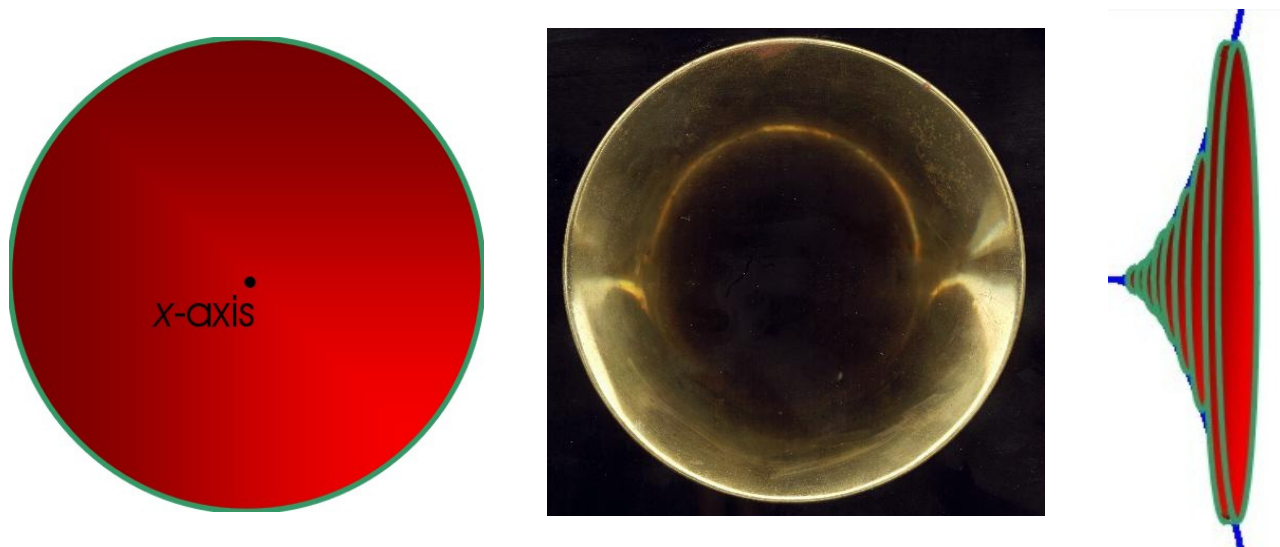
If we lay the prism onto a 3-dimensional grid, we can align it so that the length is an x -value. In cases where the width and/or height is not constant (as they are in the picture above), we'll try to express these as a function of x .

Example I: A solid is formed in the region between the functions $f(x) = x^2$ and the x -axis from $x = 0$ to $x = 2$ by rotating the curve around the x -axis. What is the volume of the resulting solid?

Consider the function $y = x^2$. If we rotate the curve, from $x = 0$ to $x = 2$, around the x -axis we would have a shape something like the bell of trumpet. (See pictures below.)



The cross-section would be a circle, with $A = \pi r^2$. At the small end, where $x = 0$, the radius would be $r = 0$, that is, the “cross-section” would have $A = 0$ and would be simply a point. At the large end, where $x = 2$, the radius of the cross-section would be $r = 2^2 = 4$.

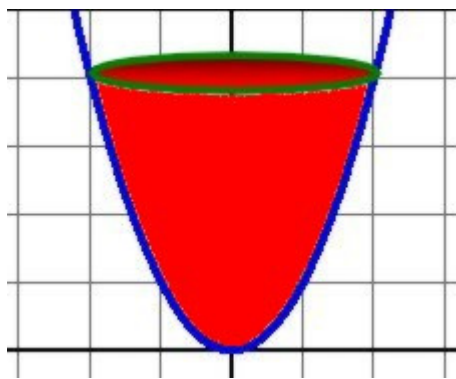
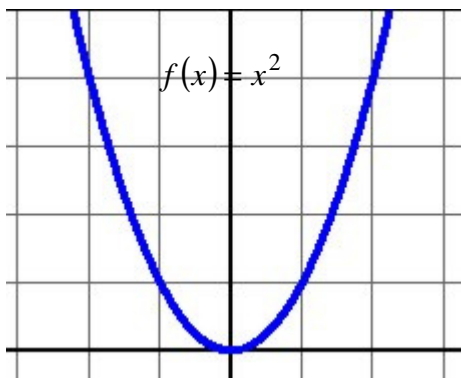


If we then create partitions along the x -axis, each partition would be a slice of our bell-shaped solid. (Think of coins with varying diameters arranged in a “sideways stack”—see picture above.) The sum of the volumes of the slices would approximate the volume of the whole. Taking the limit as the number of intervals, n , approaches infinity (i.e. size of the partitions approaches 0), the resulting Riemann sum would yield an integral.

In the specific case of $g(x) = x^2$ on the interval $0 \leq x \leq 2$, the area of the cross-section is $A = \pi(\text{radius})^2 = \pi(x^2)^2 = \pi x^4$. So the volume of the solid of revolution is

$$V = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n (\pi x_k^4) \Delta x = \int_0^2 \pi x^4 dx = \frac{\pi}{5} x^5 \Big|_0^2 = \frac{\pi}{5} (2)^5 - \frac{\pi}{5} (0)^5 = \frac{32\pi}{5}.$$

Example J: A solid is formed by taking the curve $f(x) = x^2$ from $x = 0$ to $x = 2$ and rotating it around the y -axis. What is the volume of the resulting solid? *Answer:* 8π



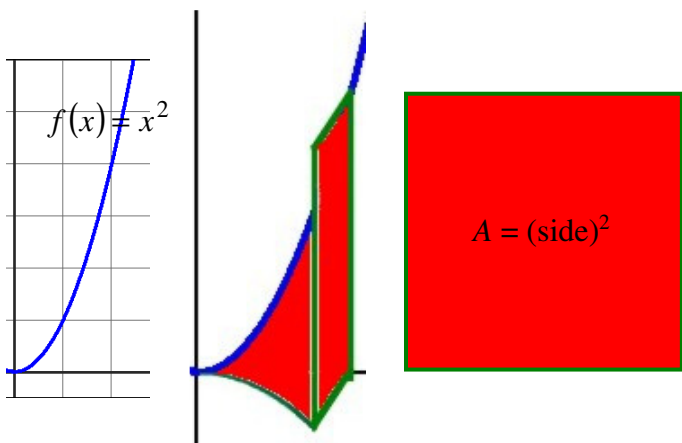
The shape of the resulting solid is something like that of a Jefferson cup, although with a wobbly bottom.

The solid whose volume we are seeking need not have a circular cross-section. The same principles as those used above yields a general formula. We create a Riemann sum of (area of cross-section) times (length).

$$V = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n A(x_k) \Delta x = \int_a^b A(x) dx \quad \text{and} \quad V = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n A(y_k) \Delta y = \int_a^b A(y) dy$$

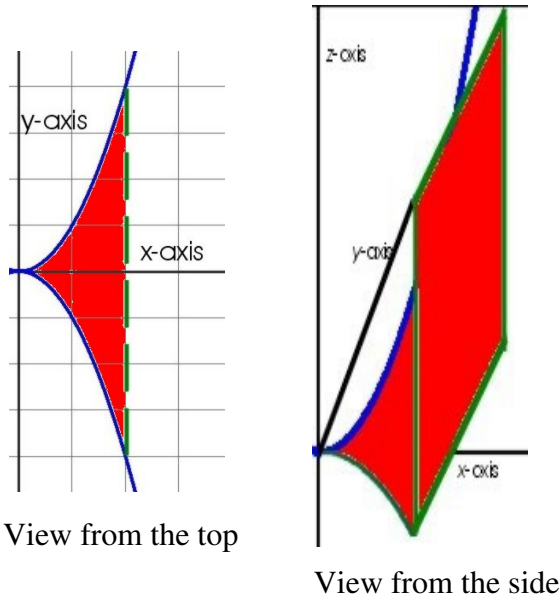
The tricky part of using the formulae above is determining how the area is to be calculated. While the circular area above used the formula πr^2 , other shapes will use their own formulae. We'll assume you already know area formulae for basic geometric shapes such as squares, triangles, etc.

Example K: A solid is formed in the region between the functions $f(x) = x^2$ and the x -axis from $x = 0$ to $x = 2$. The cross sections are squares. What is the volume of the resulting solid? *Answer:* $\frac{32}{5}$

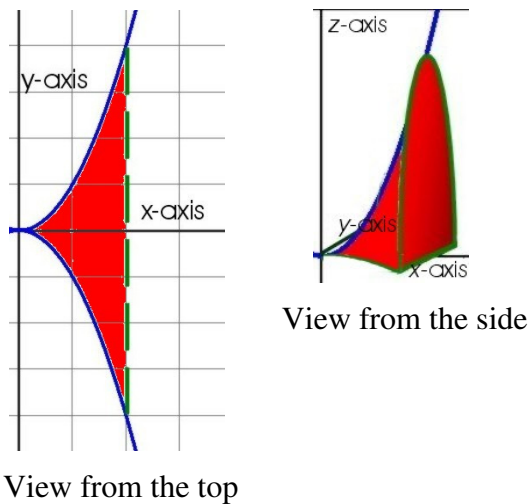


Another way to frame the same question (and the more usual way) is to say, “The base of a solid is formed by the equation $y = x^2$ from $x = 0$ to $x = 2$. The cross sections are squares. What is the volume of the resulting solid?” The visual effect would be to take the figure above, and tip it over onto its side so that the x - y plane is lying flat (horizontal) and the figure rises up (vertically) the z -axis from the base.

Example L: The base of a solid is formed by the region between the functions $f(x) = x^2$ and $g(x) = -x^2$ from $x = 0$ to $x = 2$. The cross sections are squares. What is the volume of the resulting solid? *Answer:* $\frac{128}{5}$

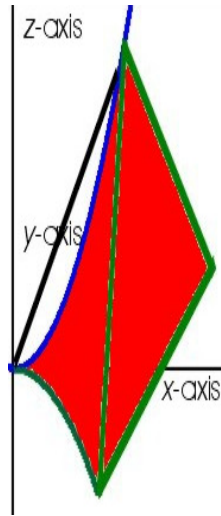
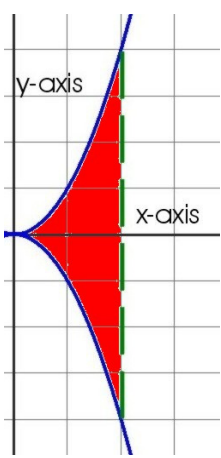


Example L extended: The base of a solid is formed by the region between the functions $f(x) = x^2$ and $g(x) = -x^2$ from $x = 0$ to $x = 2$. The cross sections are semi-circles. What is the volume of the resulting solid? *Answer:* $\frac{16\pi}{5}$



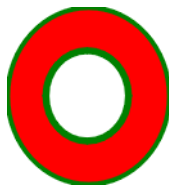
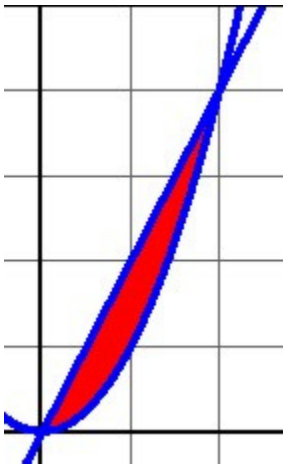
Example L extended again: The base of a solid is formed by the region between the functions $f(x) = x^2$ and $g(x) = -x^2$ from $x = 0$ to $x = 2$. The cross sections are equilateral triangles. What is the volume of the resulting solid?

Answer: $\frac{32\sqrt{3}}{5}$



View from the side

Example M: A solid is formed in the region between the functions $g(x) = 2x$ and $f(x) = x^2$ from $x = 0$ to $x = 2$ by rotating the curve around the x -axis. What is the volume of the resulting solid? Answer: $\frac{64\pi}{15}$



cross-section at $x = 1$