Calculus 141, section 6.2 Length of a Curve

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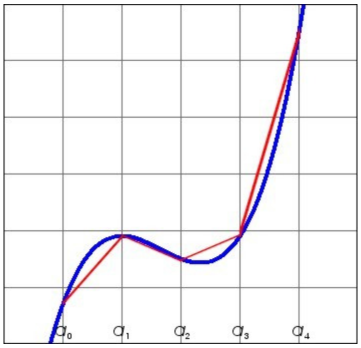
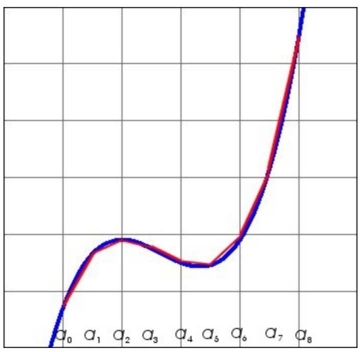


Using the same sort of mathematical thinking applied to volumes in section 6.1, the length of a curve, $f(x)$, over an interval $[a, b]$ can be approximated by a series of line segments measured over increasingly smaller intervals—the length of curve is the sum of the length of the line segments.

Split the interval $[a, b]$ into n subintervals. Then (just like when you first encountered Riemann sums) $\Delta x = \frac{b-a}{n}$. From the Pythagorean Theorem, (length of each segment) = $\sqrt{(\Delta x)^2 + (\Delta y)^2}$. From the Mean Value Theorem: there exists an x_i in each subinterval such that $\Delta y_i = f'(x_i) * (a_i - a_{i-1}) = f'(x_i) * \Delta x$. Substitution and simplification gives us a formula for the length of each line segment:

$$\text{(length of each segment)} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 + (f'(x_i) * \Delta x)^2} = \sqrt{1 + [f'(x_i)]^2} * \Delta x .$$



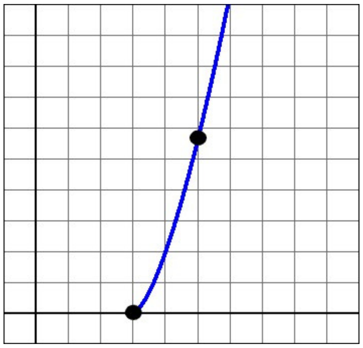
As the number of intervals, n, approaches infinity (i.e. $\| P \|$ approaches 0), the resulting Riemann sum yields a formula:

$$L = \lim_{\| P \| \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_i)]^2} * \Delta x$$

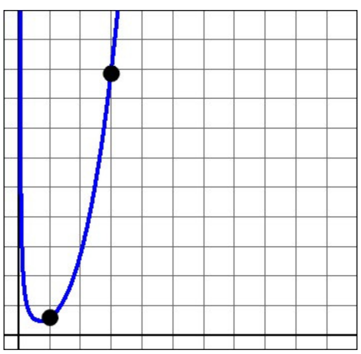
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$$

Example A: Find the length of a curve over the interval [2, 4] where $f'(x) = \sqrt{x^5 - 1}$. *Answer*: $\frac{16}{7}\left(16 - \sqrt{2}\right)$

Example B: Find the length of the curve $m(x) = 2(x-3)^{3/2}$ on the interval [3, 5]. *Answer*: $\frac{2}{27}\left(19^{3/2} - 1\right)$



Example C: Find the length of the graph of $g(x) = \dfrac{1}{3}x^3 + \dfrac{1}{4x}$ over the interval [1, 3]. *Answer*: $\frac{53}{6}$



NOTE WELL: Real life is almost never as easy as the three examples above, which is why (in upcoming sections) we will be greatly expanding our list of techniques for finding integrals.