# Calculus 141, section 6.5 Moments and Center of Gravity

notes by Tim Pilachowski

Finding the center of gravity of an object or a system might be described as summing up differential weights and equating them to simpler system—sounds elegant doesn't it? But after all, isn't that what mathematics is all about—finding a simpler way to express something complicated?

Designing the placement of the center of gravity can help prevent an SUV or boat from turning over, keep a rocket from tumbling, or minimize vibration caused by rotating parts in a machine. Designers of jet engines will suspend them from a hook to determine the "balancing point". Possibly the simplest use of center of gravity is the lever: As Archimedes is reported to have said, "ΠΑ ΒΩ ΚΑΙ ΧΑΡΙΣΤΙΩΝΙ ΤΑΝ ΓΑΝ ΚΙΝΗΣΩ ΠΑΣΑΝ," which is to say, "Give me a place to stand and with a lever I will move the whole world."

Work, in section 6.4, was defined as (force exerted) times (distance traveled). The *moment* or *moment of inertia* of a point is similarly defined as (mass) times (distance from the axis of rotation). [side note: While mass is not the same as weight, for practical purposes we will be able to use either since the objects we will consider are located in close enough proximity to each other that the difference is negligible.] The moment of a collection of points is the sum of the moments of the individual points. Specifically, the moment of a set of $n$ points about the $y$-axis is found by

$$M_y = m_1x_1 + m_2x_2 + \ldots + m_nx_n$$

Similarly, the moment of a set of $n$ points about the $x$-axis is found by

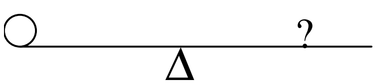
$$M_x = m_1y_1 + m_2y_2 + \ldots + m_ny_n$$

For those of you who know matrix multiplication,

$$\begin{bmatrix} m_1 & m_2 & \ldots & m_n \end{bmatrix} * \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} = \begin{bmatrix} M_x & M_y \end{bmatrix}$$

When $M_x = 0$ and/or $M_y = 0$ there is a state of equilibrium.

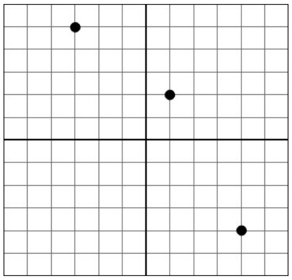
Example A: I weigh 200 lbs. My wife Sharon weighs 120 lbs. If she is 5 feet away from the center (fulcrum) of a seesaw, how far away should I be to achieve balance? *Answer*: 3 ft




The center of gravity is analogous to the mean or average from statistics: Do you remember (sum of data points) divided by (number of data points)? In finding center of gravity (also called center of mass or centroid), each point in our set may have a different mass (like homework, quizzes, exams, and final are weighted differently in calculating your average for this class), so those masses will enter into the formula in this way:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{M_y}{m} \qquad \bar{y} = \frac{m_1y_1 + m_2y_2 + \ldots + m_ny_n}{m_1 + m_2 + \ldots + m_n} = \frac{M_x}{m}$$

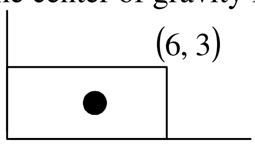
The center of gravity is the point which, if introduced into the system, will produce equilibrium.



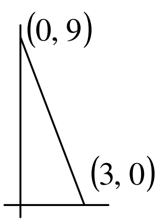
Example B: Given points (–3, 5), (1, 2) and (4, –4) with masses 1, 2, and 3 respectively, find the center of gravity. *Answer*: $(\bar{x}, \bar{y}) = \left(\frac{11}{6}, -\frac{1}{2}\right)$

Note that the center of gravity is located closer to the point with the most mass, and farther from the point with the least mass.

We now move to consideration of surfaces, i.e. a collection of an infinite number of points. For the time being, we will assume that each point on the surface has the same mass, so that the total mass can be equated with the area of the surface. Consider first the rectangle pictured below. It is intuitively clear because of the symmetry and not too difficult to show that the center of gravity is located at the physical middle.




$$(\bar{x}, \bar{y}) = \left(3, \frac{3}{2}\right)$$




What if the surface is not symmetric, for example the triangle pictured below?

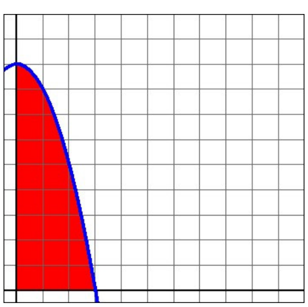
We can use the same sort of mathematical thinking applied to volumes (6.1), lengths of a curve (6.2) and work (6.4)—a series of rectangular partitions created over increasingly smaller intervals. (How convenient that rectangles are symmetric and thus easy to work with.)

The resulting integral for moment about the $y$-axis is

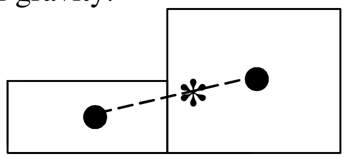
$$M_y = \int_a^b x\, f(x)\, dx$$

I will leave it to you to show that the center of gravity of the triangle above is located one-third of the way from each of the legs toward the opposite vertex.

Example C: Find the moment about the $y$-axis and $x$-coordinate of the center of gravity of the area between the curve of $f(x) = 9 - x^2$ and the $x$-axis on the interval [0, 3]. *Answers*: $\frac{81}{4}, \frac{9}{8}$



We have the moment about the $y$-axis, but what is the moment about the $x$-axis? We *could* integrate with respect to $y$, but it would get a bit messy. Instead, we'll be a bit clever. First of all, note that finding the center of gravity of a system of two surfaces (or objects such as a planet-satellite system) can be simplified to finding the center of gravity of their two respective centers of gravity.



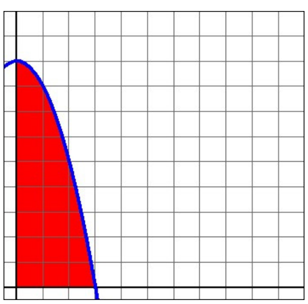
This works because, as noted above, the center of gravity is the point that produces equilibrium, i.e. its defining characteristic is that it is equal to the total of the moments of all of the points that make up the surface.

In a similar manner, instead of thinking about rotating the entire surface about the $x$-axis, we can view the process as rotating the center of gravity about the $x$-axis. Specifically, using the same rectangular partitions as for finding moment about the $y$-axis, and noting that by symmetry the center of gravity is halfway up and equal to $\frac{y}{2} = \frac{f(x)}{2}$, we can replace $x$ in the formula for $M_y$ above by $\frac{y}{2}$.

Also, noting that the area under the curve will be the same whether we integrate with respect to $x$ or to $y$, we can say that

$$M_x = \int_a^b \frac{y}{2} f(x)\,dx = \int_a^b \frac{1}{2}[f(x)]^2\,dx$$

Example C (continued): Find the moment about the $x$-axis and $y$-coordinate of the center of gravity of the area between the curve of $f(x) = 9 - x^2$ and the $x$-axis on the interval [0, 3]. *Answers*: $\frac{324}{5}$, $\frac{18}{5}$



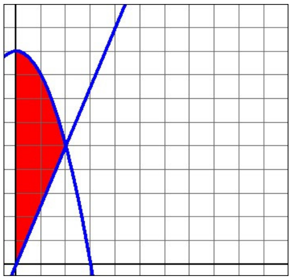
Now, what if our surface is the region between two functions? Actually, it's pretty easy. Since the area of the region is found by

$$A = \int_a^b [f(x) - g(x)]\,dx$$

it's a short hop to showing that

$$M_y = \int_a^b x\,[f(x) - g(x)]\,dx \qquad M_x = \int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx$$

Example D: Find the center of gravity of the region between $f(x) = 9 - x^2$ and $g(x) = \frac{5}{2}x$, for $x \geq 0$.



*Answer*: $\left(\frac{22}{31}, \frac{778}{155}\right)$

When comparing the center of gravity for Example C with the center of gravity for Example D, note that the removal of surface from the lower right has caused the center of gravity to move to the left and up from where it was before, i.e. away from the decrease and toward the remaining mass.

One more thing you'll need for the practice questions and WebAssign: the Theorem of Pappus and Guldin. If we revolve a two-dimensional region (think cross-section) with area = $A$ around a line $l$, a three-dimensional space is created. If we think of the 3-D space produced as a "circle" with radius = $b$ = the distance from $l$ to the center of gravity of the region (cross-section), then the volume of the 3-D space would be like the "circumference" times "area of the cross-section". Thus, we get $V = 2\pi bA$. Note that using this formula requires that we have already found the location of the center of gravity of the revolving region.