Calculus 141, section 6.8 Parametric Equations, Length & Surface Area
notes by Tim Pilachowski

Do you remember the formula for length of a curve derived in section 6.2:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \quad ?$$

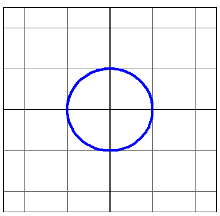
The formula for length of a curve C defined by parametric functions is similar, and yields a similar result:

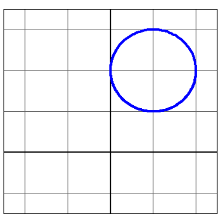
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Note that if C is defined by $x = t$ and $y = f(t)$, we would have almost the exact same thing as the 6.2 formula, but in terms of t rather than x. Note also that while C can often be parametrized in more than one way, each set of parametric equations, when inserted into the formula, will yield the same result. That is, the length of a curve is not dependent upon which parametric equations are used.

Example (1) [a) Lecture 6.7 the unit circle, b) Lecture 6.7 Example D]

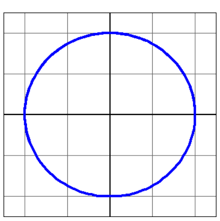
$P(t) = (\cos t, \sin t)$ and $P(t) = (1 + \cos t, 2 - \sin t)$ for $0 \le t \le 2\pi$ *Answers*: 2π, 2π





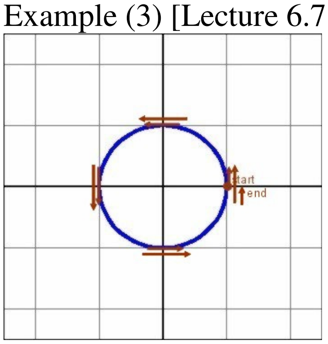
Example (2) [Lecture 6.7 Example A] Example A from lecture notes for section 6.7: $x = 2\cos t$ and $y = 2\sin t$ for t moving from 0 to 2π.

Answer: 4π



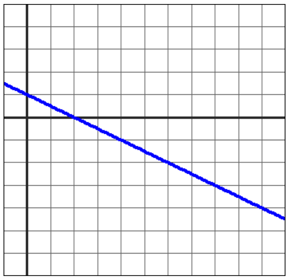
Example (3) [Lecture 6.7 Example B]: $x = \cos(2t)$ and $y = \sin(2t)$ for t moving from 0 to 2π.

Answer: 4π



Since an ellipse can be expressed parametrically as $x = a\cos(t)$ and $y = b\sin(t)$, the length formula is useful in proving a geometric formula for the circumference of an ellipse, but requires integration techniques we don't have yet.

Example (4) [Lecture 6.7 Example E]: Given the parametric equations $x = 2t$ and $y = 1 - t$, find the length of the curve from $t = 0$ to $t = 5$. *Answer*: $5\sqrt{5}$



The formula for surface area developed in the text relies on work done in section 6.3, which is not included in our syllabus. The basic principle involves a curve C, expressed parametrically, rotated about the x-axis. You might think of it as the length formula used above given a circumference of $2\pi r$ by the rotation. The formula for surface area of the rotated curve C is:

$$S = \int_a^b 2\pi\, g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\; dt$$

Example (5) [Lecture 6.7 Example A revised]: Find the surface area of the figure obtained by rotating the graph of $x = 2\cos t$ and $y = 2\sin t$ for t moving from 0 to π. *Answer*: 16π