Calculus 141, section 7.1 InverseFunctions

notes by Tim Pilachowski

Given a function f and its inverse, f^{-1}, the following will always be true:

1. If $f(a) = b$, then $f^{-1}(b) = a$ (This fact and the statement in point #2 below is actually the same information.)

2. If (a, b) is a point on the graph of f, then (b, a) will be on the graph of f^{-1}. You can use this fact to use the graph of a function to sketch the graph of its inverse: locate two or three points on the graph of f, swap coordinates, plot the new points, connect the dots, and you have the graph of f^{-1}. (See example A below.)

3. The domain of f = the range of f^{-1}, and the range of f = the domain of f^{-1}.

4. The graph of f and the graph of f^{-1} are symmetric with respect to the line $y = x$.

5. $f \circ f^{-1} = x$ and $f^{-1} \circ f = x$. To show that two functions are inverses you must do *both* compositions.

Question: What common function is its own inverse? The answer next time.

Example A: Does the function $f(x) = \sqrt{x-2}$ have an inverse? If so, what does it look like?
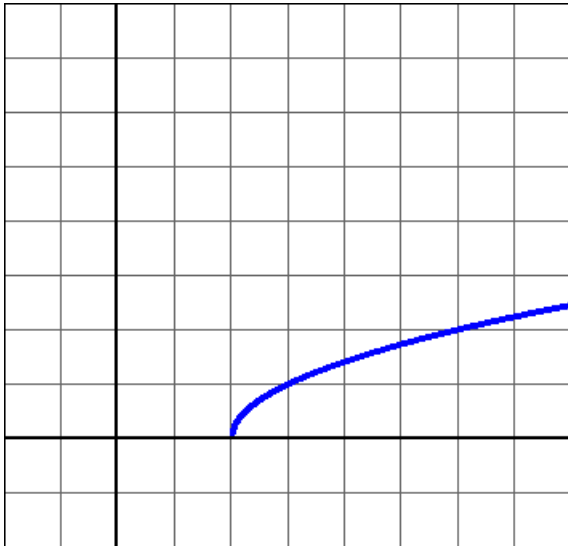
Important note: It is only because the range of f, and therefore the domain of f^{-1}, is restricted to non-negative numbers that we can say that the absolute value of x is x, since the absolute value of a non-negative number is equal to the number itself.

Given a function has an inverse, can it be found? You probably remember something like this from Precalculus—Using items #1 through #3 above, we can develop a process to find the inverse of a function.

Example B: Given $h(x) = x^3 + 5$, find the equation of its inverse. *Answer*: $h^{-1}(x) = \sqrt[3]{x-5}$

In general, the inverse function is not usually easy to find or specify.

One way to tell whether on not a function has an inverse is by applying the "horizontal line test" to the graph: If any horizontal line passes through only one point of the curve, then the function has an inverse.
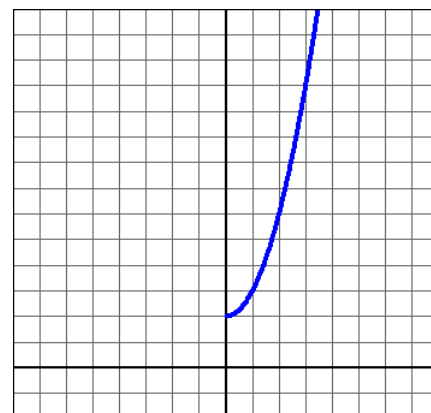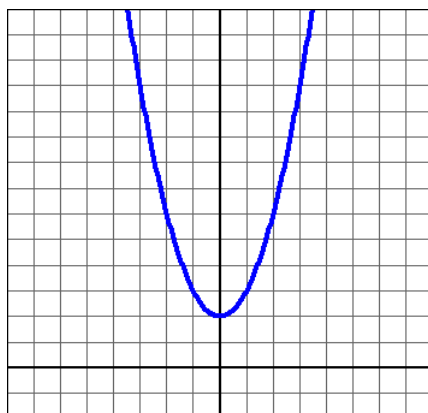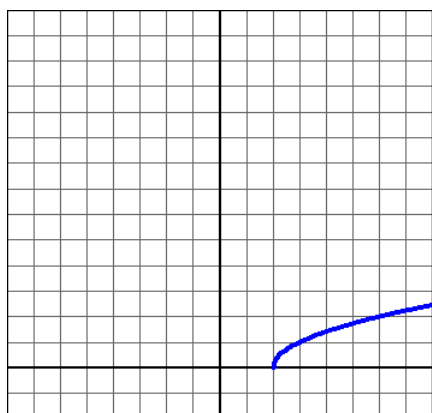
Why does the horizontal line test work? Because to have an inverse a function must be **one-to-one**: each value x in the domain of f leads to exactly one value of y in the range. If an x in the range led to more than one y in the range, the "inverse" could not be a function! More formally, a function is one-to-one and has an inverse if and only if $a \neq b$ implies $f(a) \neq f(b)$.

While the horizontal line test is useful in determining when a function does *not* have an inverse, it is not sufficient evidence to prove that a function *has* an inverse. However, we can use calculus techniques to determine the existence of an inverse: If a function is strictly increasing or strictly decreasing (i.e. has no relative maximum or minimum) then it will be one-to-one and will have an inverse. Derivatives enable us to determine the existence of relative maxima/minima. If the domain is restricted to an interval (like the third graph above), and the function is either strictly increasing or strictly decreasing on that interval, then we'll say that the function has an inverse on that interval.

Example C: Find the largest interval containing $x = 0$ on which $g(x) = x^3 + \dfrac{3}{2}x^2 - 6x + 1$ has an inverse.

Answer: [–2, 1]

In addition, since g is continuous from $g(-2) = 11$ to $g(1) = -5/2$, we know that g^{-1} is continuous on the interval $[-5/2, 11]$.

Let's assume a function f is differentiable.
If it has an inverse, is the inverse differentiable?
If the inverse is differentiable, is there a relationship between the two derivatives?

Reconsider Example A above. At (6, 2) f is increasing slowly. On the other hand, at (2, 6) f^{-1} is increasing quickly. Theorem 7.5 formalizes this intuitive relationship: For a function f which is continuous on a given open interval: if $f(a)=c$, and $f'(a)$ exists, and $f'(a)\neq 0$, then $f^{-1}(c)=a$ and $\left(f^{-1}\right)'(c)=\dfrac{1}{f'(a)}$.

The text proves Theorem 7.5 using the definition of the derivative.

Example D. Given $f(x)=3x+6$, first find $f^{-1}(x)$, then show that $\left(f^{-1}\right)'(x)=\dfrac{1}{f'(x)}$ for all x.

Example A revisited. Given $f(x)=\sqrt{x-2}$ and $f^{-1}(x)=x^2+2$, $x\geq 0$, show that $\left(f^{-1}\right)'(2)=\dfrac{1}{f'(6)}$.

Example B revisited. Given $h(x)=x^3+5$, does $h'(1)=\dfrac{1}{\left(h^{-1}\right)'(6)}$? *Answer*: yes

Example C revisited: Given $g(x)=x^3+\dfrac{3}{2}x^2-6x+1$, first find the value of c for which $g(-1)=c$, then determine $g^{-1}(c)$ and $\left(g^{-1}\right)'(c)$. *Answers*: $\dfrac{15}{2}$, -1, $-\dfrac{1}{6}$

Side note: The last question in WebAssign is a review of a topic from section 5.7 that we'll need in the near future.