Calculus 141, section 7.3 Logarithmic Functions

notes by Tim Pilachowski

Since e^x and $\ln(x)$ differentiate and integrate so nicely, wouldn't it be terrific if we could get the more general b^x and $\log_b(x)$ to do the same? Actually, if we define b^x in terms of e and define $\log_b(x)$ in terms of $\ln(x)$, it makes it pretty close to simple.

Recall that e^x and $\ln x$ are inverse functions. So, given a number b it will be true that $b = e^{\ln b}$.
Also, $b^x = \left(e^{\ln b}\right)^x = e^{\ln b * x}$. Specifically, we will define

$$b^x = e^{x \ln b} \quad \text{and} \quad \log_b x = \frac{\ln x}{\ln b}.$$

We can use what we already know about derivatives and chain rule to work with exponentials and logarithms in any base. But first, we have to ascertain whether these definitions contradict anything we already know about exponentials and logarithms and their properties. It isn't very difficult to show (and the text does a good job of showing) that all of the exponential properties hold true:

$$b^0 = 1, \quad b^1 = b, \quad b^{m+n} = b^m * b^n, \quad b^{-n} = \frac{1}{b^n}, \quad \left(b^m\right)^n = b^{m*n}.$$

and that the logarithm properties hold true as well:

$$\log_b\left(m^n\right) = n \log_b m \quad \text{and} \quad \log_b(m*n) = \log_b m + \log_b n.$$

But let's get to the next level: finding derivatives of exponential function. To do so, use the Chain Rule.

$$\frac{d}{dx}\left(b^x\right) = \frac{d}{dx}\left(e^{x \ln b}\right) = \ln b * e^{x \ln b} = \ln b * \left(e^{\ln b}\right)^x = \ln b * b^x$$

Don't forget that since the base b will be a designated numeric value, then $\ln b$ will also be a numeric value, and *not* a function of a variable. Side note: You could memorize this formula, but alternately you could just remember to convert your exponential using $b^x = e^{x \ln b}$ and differentiate using the Chain Rule.

Example A: Verify the rule for differentiation of a power function $y = x^r$.

Example B: Differentiate $f(x) = x^2 * 3^{5x}$. *Answer*: $3^{5x}\left(2x + 5(\ln 3) * x^2\right)$ *or* $3^{5x}\left(2x + 5\ln(3) * x^2\right)$

We can also use the above definition with the Chain Rule to differentiate an exponential in which both the base and the exponent are functions, i.e. of the form $p(x)^{\,q(x)}$.

Example C: Find the first derivative of $f(x)=\left(x^2+1\right)^{x-1}$. *Answer*: $\left[\ln\left(x^2+1\right)+\frac{2x(x-1)}{x^2+1}\right]*e^{(x-1)*\ln\left(x^2+1\right)}$

How about finding the derivative of a logarithm in a base other than e?

Example D: Find a generic formula for the first derivative of a logarithm in base b. *Answer*: $\frac{1}{x\ln b}$

Now that we can find derivatives, can we integrate exponentials and logarithms? But, of course!

$$\frac{d}{dx}\left(b^x\right)=\ln b*b^x \Rightarrow \frac{1}{\ln b}*\frac{d}{dx}\left(b^x\right)=\frac{d}{dx}\left(\frac{1}{\ln b}*b^x\right)=b^x \Rightarrow \int b^x=\int\frac{d}{dx}\left(\frac{1}{\ln b}*b^x\right)dx=\frac{1}{\ln b}b^x+C$$

Example E: Find $\int x^4*3^{-x^5}\,dx$. *Answer*: $-\frac{1}{5\ln 3}\left(3^{-x^5}\right)+C$

Example F: Evaluate $\int_1^3\frac{\log_3 x}{x}\,dx$. *Answer*: $\frac{\ln 3}{2}$