Calculus 141, section 7.5 Inverse Trigonometric Functions

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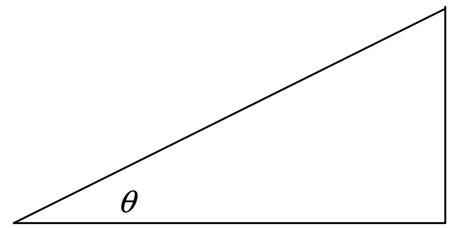
If you haven't gone back to review the basics of trig functions, now is a good time to do so. One method to remember values is based on the unit circle. Another uses ratios from a right triangle.

t	$\cos t$	$\sin t$
0	$\sqrt{4}/2 = 1$	$\sqrt{0}/2 = 0$
$\pi/6$	$\sqrt{3}/2$	$\sqrt{1}/2 = 1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{1}/2 = 1/2$	$\sqrt{3}/2$
$\pi/2$	$\sqrt{0}/2 = 0$	$\sqrt{4}/2 = 1$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$



Now, since the trig functions, like all periodic functions, are not one-to-one, we can only talk about them having an inverse on an interval. Specifically, we need an interval on which the function is strictly increasing or decreasing (cf. section 7.1, theorem 7.5). For sine, the interval [– π/2, π/2], with a range of [– 1, 1], satisfies this condition. Thus we can say:

$$\sin^{-1} x = y \text{ if and only if } \sin y = x \quad \text{domain} = [-1, 1] \quad \text{range} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin^{-1}(\sin x) = x \text{ for } x \text{ on the interval } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \sin\left(\sin^{-1} x\right) = x \text{ for } x \text{ on the interval } [-1, 1]$$

In words, $\sin^{-1} x$ is the unique angle between – π/2 and π/2 whose sine is x. I'll use the notation $\sin^{-1}$ consistently, although "arcsin" is just as common. WebAssign uses "asin( )".

Example A: Find $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$ *Answers*: $\frac{\sqrt{2}}{2}$; $-\frac{\pi}{4}$

Since cosine is decreasing on the interval [0, π], we can define

$$\cos^{-1} x = y \text{ if and only if } \cos y = x \quad \text{domain} = [-1, 1] \quad \text{range} = [0, \pi]$$

$$\cos^{-1}(\cos x) = x \text{ for } x \text{ on the interval } [0, \pi] \qquad \cos\left(\cos^{-1} x\right) = x \text{ for } x \text{ on the interval } [-1, 1]$$

Example B: Evaluate $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$. *Answer*: $\frac{3}{5}$

Note that the range of $\cos^{-1}$ implies that the angle is in Quadrant I. An alternate method would, of course, be to use the identity $\sin^2 x + \cos^2 x = 1$. Once you have determined sine and cosine, any of the other trig functions can be found using its definition.

For tangent, an interval between asymptotes is convenient, since tangent is increasing on any of these.

$$\tan^{-1} x = y \text{ if and only if } \tan y = x \quad \text{domain} = (-\infty, \infty) \quad \text{range} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\tan^{-1}(\tan x) = x \text{ for } x \text{ on the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad \tan\left(\tan^{-1} x\right) = x \text{ for } x \text{ on the interval } (-\infty, \infty)$$

When called upon to evaluate one of the other inverse trig functions, you can think in terms of reciprocals.

Example C: Determine $\sin\left(\sec^{-1}\left(-\sqrt{2}\right)\right)$. *Answer*: $\frac{\sqrt{2}}{2}$

The inverse trig functions can be differentiated:

$$y = \sin^{-1} x \to \sin y = x \to \frac{d}{dx}\sin y = \frac{d}{dx}x \to \text{via implicit differentiation } \cos y\frac{dy}{dx} = 1 \to \frac{dy}{dx} = \frac{1}{\cos y}$$

Note that on the (open interval) range of $\sin^{-1} x$, $(-\pi/2, \pi/2)$, $\cos y > 0$, and we can substitute

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Note that the derivative of $\sin^{-1} x$ is not defined for $x = -1$, nor for $x = 1$.

Example D: Find the first derivative of $f(x) = \sin^{-1} x^2$. *Answer*: $\frac{2x}{\sqrt{1-x^4}}$

The derivatives of $\tan^{-1} x$ and $\sec^{-1} x$ can be found using a similar process to the one used above for $\sin^{-1} x$.

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1} \qquad \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$$

Similarly we can derive the following.

$$\frac{d}{dx}\cos^{-1} x = -\frac{d}{dx}\sin^{-1} x \qquad \frac{d}{dx}\cot^{-1} x = -\frac{d}{dx}\tan^{-1} x \qquad \frac{d}{dx}\csc^{-1} x = -\frac{d}{dx}\sec^{-1} x$$

Results are not really a surprise given the identity $\sin x = \cos(\pi/2 - x)$ meaning that

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \text{for all } x \text{ in } [-1, 1]$$

The corresponding integral identities are

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}+C = -\cos^{-1}\frac{x}{a}+C \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C = -\frac{1}{a}\cot^{-1}\frac{x}{a}+C$$

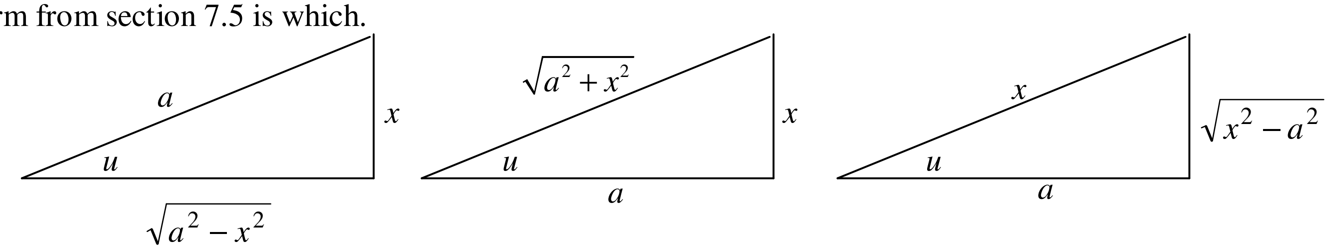
$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+C = -\frac{1}{a}\csc^{-1}\frac{x}{a}+C$$

The text does examples of using completing the square to rewrite an integral into an equivalent form so that the formulae above can be used to get $\sin^{-1}$ (examples 4 & 5) and $\tan^{-1}$ (examples 8 & 9). These two formulae are the only ones you'll need for the practice exercises and for WebAssign. I'm going to tackle $\sec^{-1}$ as an illustration of both completing the square and substitution.

Example E: Evaluate $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x-5}}$. *Answer*: $\frac{1}{3}\sec^{-1}\left(\frac{x-2}{3}\right)+C$

A look ahead to section 8.3:

The Pythagorean formula and triangle ratios from Trigonometry gives us a way to remember which inverse trig form from section 7.5 is which.



1) $\sin u = \dfrac{x}{a}\left(=\dfrac{opp}{hyp}\right)$ $\qquad$ 2) $\tan u = \dfrac{x}{a}\left(=\dfrac{opp}{adj}\right)$ $\qquad$ 3) $\sec u = \dfrac{x}{a}\left(=\dfrac{1}{\cos u}=\dfrac{hyp}{adj}\right)$

$u = \sin^{-1}\left(\dfrac{x}{a}\right)$ $\qquad$ $u = \tan^{-1}\left(\dfrac{x}{a}\right)$ $\qquad$ $u = \sec^{-1}\left(\dfrac{x}{a}\right)$