Calculus 141, section 7.6 L'Hôpital's Rule

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Division by zero is undefined, and a denominator of zero presents a tricky scenario when evaluating limits for fractions in which the numerator also approaches zero (the "indeterminate form $\frac{0}{0}$"). L'Hôpital's Rule, discovered by Bernoulli, provides us a way to accomplish this using derivatives:

$$\lim_{x \to *} \frac{f(x)}{g(x)} = \lim_{x \to *} \frac{f'(x)}{g'(x)} \text{ where } \lim_{x \to *} f(x) = 0, \lim_{x \to *} g(x) = 0, \; g'(x) \neq 0 \text{ for } x \text{ near } *, \text{ and } \lim_{x \to *} \frac{f'(x)}{g'(x)} \text{ exists.}$$

In this rule, * can be replaced by a, a^-, a^+, ∞ or $-\infty$.

The text offers a proof using the Quotient Rule for Limits. From a graphical sense, If $f(x)$ and $g(x)$ are differentiable near a, their graphs will be locally linear and their derivatives will represent the linear slope:

$$\frac{f(x)}{g(x)} = \frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} \to \frac{m_f}{m_g}, \text{ and as } x \to a, \; \frac{m_f}{m_g} \to \frac{f'(x)}{g'(x)}.$$

We don't have time for the full proof of L'Hôpital's Rule—you'll have to go to the Appendix. Just a quick note: For some textbook practice problems and WebAssign you'll need to use the recent work we've done with exponentials, logarithms and inverse trig functions (see Example 3).

In the next few examples, note that the limit of both numerator and denominator is zero, and that L'Hôpital's Rule applies.

Example A: Find $\lim_{x \to 0} \frac{\sin^{-1}(x)}{e^x - 1}$. *Answer*: 1

Example B: Evaluate $\lim_{x \to 0^+} \frac{\sin x}{x^2}$ and $\lim_{x \to 0^-} \frac{\sin x}{x^2}$. *Answer*: ∞; $-\infty$

Example C: Find $\lim_{x \to 0} \frac{\sqrt{x+1} - 1 - \frac{x}{2}}{x^2}$. *Answer*: $-\frac{1}{8}$

Example D: Evaluate $\lim\limits_{x \to \infty} \dfrac{x}{2^x}$.

Example E: Find $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \csc x \right)$. *Answer*: 0

Example F: Evaluate $\lim\limits_{x \to \infty} x \sin\left(\dfrac{1}{x} \right)$. *Answer*: 1

Example G: Find $\lim\limits_{x \to 0^+} x^{\sin x}$. *Answer*: 1

I'll refer you to the text for other, similar examples, including $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x} \right)^x = e$.

Examples H: Consider $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{\ln x}$ and $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{x - \cos x}$. *Answer*: can't do; 1