Calculus 141, section 7.7 Differential Equations (Introduction)

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You know how to differentiate a number of types of equations:

power rule: $f(x) = ax^5 \;\Rightarrow\; f'(x) = 5ax^4$

chain rule: $g(x) = e^{\sin t} \;\Rightarrow\; \dfrac{dx}{dt} = \cos t\, e^{\sin t} \;\Rightarrow\; \dfrac{d^2x}{dt^2} = -\sin t\, e^{\sin t} + \cos t\left(\cos t\, e^{\sin t}\right) = e^{\sin t}\left(-\sin t + \cos^2 t\right)$

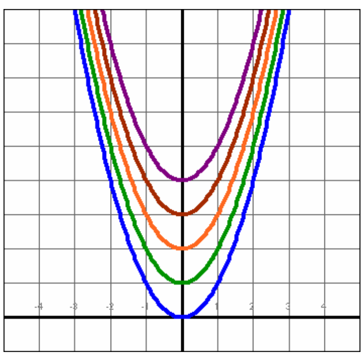
implicit differentiation (For full details see Lecture 7.2 Example B.):

$$e^{xy} = 2x + y \;\Rightarrow\; \left(y + x\frac{dy}{dx}\right)e^{xy} = 2 + \frac{dy}{dx} \;\Rightarrow\; xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} = 2 - ye^{xy} \;\Rightarrow\; \frac{2 - ye^{xy}}{xe^{xy} - 1}$$

In some cases we can (and did) integrate to move from a derivative back to the original function. In other cases, the process of solving the differential equation (DE) is more involved.

Terminology notes: The ***order*** of a differential equation is equal to the highest-order derivative involved. So $y' = 2y - t$ is a first-order DE, and $y'' = 3y' + 2y - t$ is a second-order DE.

Solutions that look like $y = ax^5$ and $y = e^{\sin t}$ are called ***explicit*** solutions. Solutions that look like $e^{xy} = 2x + y$ are called ***implicit*** solutions.



Usually a DE will have many solution functions. Think about the integration we've done up to this point: You had to remember to write "+ C" as part of your answer. Each possible value for the constant C (which in some applications is also called a parameter) represents a different function. (Note that the placement of the constants in your answers will not always be the same!) Solutions that look like $y = x^2 + C$ are called ***general*** solutions, because one can derive all solutions by substituting an appropriate value for the constant a (also sometimes called a parameter) in a given situation. A solution that looks like $y = x^2 + 3$ is called a ***particular*** solution because there are no parameters which will change value in varied situations. An initial condition adds a point of reference, allowing us to apply the general solution to a particular instance.

(A cautionary note: While it is sometimes tempting to do so, "initial condition" should not be understood to always mean "when *time* = 0".) To further illustrate:

$h(t) = -16t^2 + v_0 t + h_0$ is a *general* formula for height of an object reacting to the force of Earth's gravity

$h(t) = -16t^2 + 5t + 40$ is a *particular* formula for an object with initial velocity = 5 and initial height = 40

Example A: Solve $\dfrac{dy}{dx} = y$. *Answer*: $f(x) = ce^x$

Example A extended: Solve $\frac{dy}{dx} = y$ given the initial condition $f(0) = 12$. *Answer*: $f(x) = 12e^x$

While many of the differential equations you encounter in this class will seem to be somewhat capricious, in the real world differential equations have many useful applications. Observations about how something changes (grows or declines) can lead researchers to an equation which describes its behavior.

In this section, you'll be asked to accomplish several types of tasks:

Example B: Determine whether $y = a \sin x$ is a general solution to the second order DE $y'' = -y$. *Answer*: yes

Example B extended: Is $y = a \sin x + b \cos x$ is a general solution to the second order DE $y'' = -y$. *Answer*: yes

Example C: Show that $y = \frac{3}{2}e^{t^2} - \frac{1}{2}$ is a solution of the DE $y' - 2ty = t$.

Example D: Verify that $y = \sqrt{x}$ satisfies the conditions $y' = \frac{1}{2y}$ and $y(1) = 1$.

Example E: Consider the DE $y^2 y' = 3t^2$. Verify that $y = \sqrt[3]{3t^3 + C}$ is a general solution.

Example E extended: Find C such that $y = \sqrt[3]{3t^3 + C}$ is a particular solution to $y^2 y' = 3t^2$ for which $y'(1) = \frac{1}{3}$.

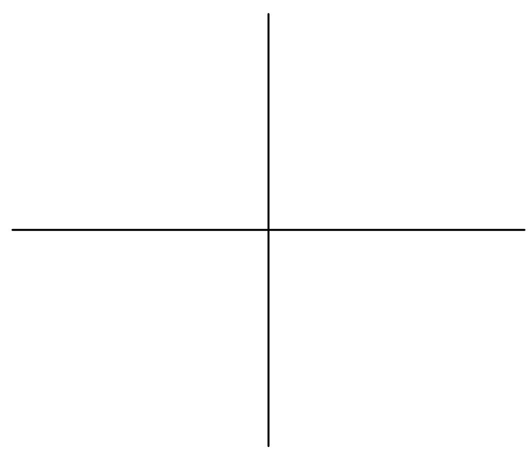
Note that there are two possible answers here: 24 and – 30, generated when you take both sides of the equation to the 3/2 power, and remember to include both the + and – possibilities when taking the square root.

You will also be asked to plot a “direction field” of a differential equation. For a grid of points (x, y), plug these values into the derivative formula. Draw short line segment (slope segment) at that point to represent the derivative, i.e. the instantaneous slope evaluated at that point. The revealed patterns of connected line segments represent shapes of possible particular solution equations, each with its own initial conditions.

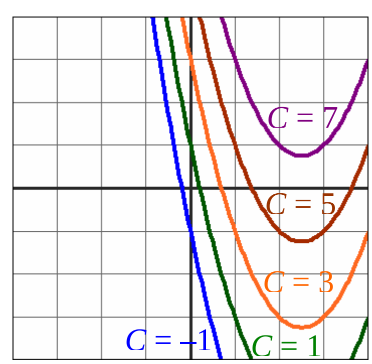
Example F: Plot the direction field of the DE $y' = 2t - 5$.

A table of values gives us the slope segments to plot (y values are given vertically; t values horizontally). Note that these “slopes” vary only in relation to t.

	– 3	**– 2**	**– 1**	**0**	**1**	**2**	**3**
3							
2							
1							
0							
– 1							
– 2							
– 3							



Solving $y' = 2t + 5$ by anti-differentiation gives us $y = t^2 - 5t + C$. Note that the pattern of slope segments forms the shape of a parabola with vertex at $x = \frac{5}{2}$. The figure to the right gives graphs of our solution for various values of C.



Example G: What if the derivative varies in relation to both *t* and *y*? Consider $y' = y + t$. Note that y' becomes increasingly negative as either *t* or *y* becomes more negative, equals 0 where *y* = – *t*, and becomes increasingly positive as either *t* or *y* becomes more positive.

	– 6	**– 4**	**– 2**	**0**	**2**	**4**	**6**
12							
8							
4							
0							
– 4							
– 8							
– 12