Calculus 141, section 7.8 Differential Equations (Methods)

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Section 7.7 introduced differential equations (DEs). Section 7.8 provides two methods for solving DEs. The first type is a separable differential equation, i.e. one in which we can separate elements containing the *x* variable from ones containing the *y* variable.

Separable DEs will generally have one of two forms:

$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{dy}{g(y)} = f(x)\,dx \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow g(y)\,dy = f(x)\,dx$$

Example A: The temperature *y* of a cup of coffee changes with time (Newton's Law of Cooling). Find an equation to express temperature as a function of time. *Answer*: $y = Ke^{-0.1t} + 70$

We can easily check this general solution using the methods of section 7.7. Note several things:
1) The coefficient –0.1 is a "constant of proportionality"specific to this particular coffee cup, and would have been found via observation.
2) Domain of ln (y – 70), normally a concern, is not a worry in this case since the temperature of the coffee will not go below room temperature. Indeed, our equation for *y* has a horizontal asymptote at $y = 70$.
3) While each of the two integrals would usually be written with a "+ C", since these are arbitrary constants the two can be combined on one side of the equation.
4) We could leave the equation with e^C as the coefficient, but again since C is arbitrary, e^C will itself be a constant and has been written more simply as K.

Example A extended: If a cup of coffee in a 70º room began at 190º, what is its temperature after 5 minutes?
Answer: $y(5) = 120e^{-0.5} + 70 \cong 142.78°$ F

Example B: Solve $y' = \frac{3t^2}{y^2}$. *Answer*: $y = \sqrt[3]{3t^3 + C}$

We already checked this result—see lecture notes 7.7 Example E—and found the particular solution for which $y' = \frac{1}{3}$. Note that since C_1 is an arbitrary constant, we easily and legally replaced $3 * C_1$ with a more generic C.

This method may seem pretty cavalier—we're almost treating $\frac{dy}{dt}$ as if it is a fraction, which of course it is not. Is our procedure legal? Of course, and we can justify it using substitution. When we have $f(y)\frac{dy}{dt} = g(t)$, we take the integral of both sides with respect to t. Letting $u = y$, then $du = \frac{dy}{dt}\,dt$, and thus $\int f(y)\frac{dy}{dt}\,dt = \int f(u)\,du = \int f(y)\,dy$.

Example C: Solve $y' - 2ty = t$. *Answer*: $y = Ce^{t^2} - \frac{1}{2}$

Once again a complicated constant was replaced with the more generic C. Also note that the particular solution we checked in section 7.7 lecture notes Example C fit this solution.

Example D: Solve $y' = (1 + y^2)e^x$. *Answer*: $y = \tan(e^x + C)$

Note how, while substitution helped solve the integral in Example C, it would not have worked here. We had to recognize $\tan^{-1}$.

We now turn to a more difficult scenario: linear first-order differential equations. These are of the form

$$\frac{dy}{dx} + P(x) * y = Q(x)$$

To solve these, we need to do some clever things. First we'll define $S(x) = \int P(x)\,dx$, i.e. an anti-derivative.

Next, multiply both sides of the DE by $e^{S(x)}$ and integrate with respect to *x*.

$$e^{S(x)}\frac{dy}{dx} \quad + \quad P(x)\,e^{S(x)}y \quad = \quad e^{S(x)}Q(x)$$

$$\int e^{S(x)}\frac{dy}{dx}\,dx + \int P(x)\,e^{S(x)}y\,dx = \int e^{S(x)}Q(x)\,dx$$

$$\int f \quad g' \quad + \quad f' \quad g \;\; dx =$$

$$f \;\; g =$$

$$e^{S(x)}y = \int e^{S(x)}Q(x)\,dx$$

$$y = e^{-S(x)}\int e^{S(x)}Q(x)\,dx$$

In using this formula, it will be important to correctly identify *P* and *Q*, and also to correctly construct the ***integrating factor*** $e^{S(x)}$. It is called an integrating factor because it transforms the left-hand side into a form that can be integrated because it is the result of differentiating by the Product Rule.

Example E: Solve $x\,y' = x^2 + 3y, \;\; x > 0$. *Answer*: $y = -x^2 + Cx^3, \quad x > 0$

Note that *S* can be any anti-derivative, so we might as well pick the simplest one and make our life a little easier.

Also note, that instead of using a memorized formula $y = e^{-S(x)}\int e^{S(x)}Q(x)\,dx$, I wrote the left-hand side as the antiderivative of a product of functions. Feel free to use whichever method works for you.

Example E extended: Given the initial condition $y(1) = 2$, find the particular solution. *Answer*: $y = 3x^3 - x^2$

Example F: An anti-coagulant is introduced intravenously at a rate of 0.5 mg per hour. At the same time, 2% of the drug in the bloodstream is absorbed into the body. Solve $y' = -0.02y + 0.5$ to get an equation describing the amount of anti-coagulant in the bloodstream. *Answer*: $25 + Ce^{-0.02t}$

Examples G: $xy' = x^2 + y^2$ is not linear, and $xy'' = x^2 + y$ is not first-order, so we cannot use the integrating factor method for either of these.

Bonus Example H: Solve $y' - y^3 \cos t = 0$.
This one is easily separable.

$$y' = y^3 \cos t \;\Rightarrow\; y^{-3} \frac{dy}{dt} = \cos t \;\Rightarrow\; \int y^{-3}\, dy = \int \cos t\, dt \;\Rightarrow\; -\frac{1}{2} y^{-2} = \sin t \;\Rightarrow\; -\frac{1}{2\sin t} + C = y^2$$

$$\Rightarrow\; y = \pm\sqrt{\frac{1}{2\sin t} + C}$$

Notes: The general solution is actually two functions, the “plus” and “minus” versions. The correct one would be determined by initial conditions, when available.

Bonus Example I: Solve $y' + 2ty = 6t$.
This is already in the standard form for a linear first-order DE.
$P(t) = 2t$ and $Q(t) = 6t$. Then $S(t) = \int 2t\, dt = t^2$ and the integrating factor is e^{t^2}.

$$e^{t^2} y' + 2te^{t^2} y = 6te^{t^2} \;\Rightarrow\; \int \frac{d}{dt}\left[e^{t^2} y\right] dt = \int 6te^{t^2}\, dt \;\Rightarrow\; u = t^2,\, du = 2t\, dt \;\Rightarrow\; e^{t^2} y = 3\int e^u\, du$$

$$\Rightarrow\; e^{t^2} y = 3e^{t^2} + C \;\Rightarrow\; y = 3 + Ce^{-t^2}$$