Calculus 141, section 8.1 Integration by Parts

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In an integral such as $\int t \sin(6t)\, dt$, substitution won't work. While we can identify $6t$ as an "inside" function, and its derivative as 6, there is nothing in either u or du to substitute for the t which is located "outside" the sine. A clue as to how to approach this integral is contained in the fact that it is a product of two functions. Can we find a way to "work backward" from the product rule? The answer is "Yes", and it's called *integration by parts*.

Recall the product rule: $\frac{d}{dx}[u * v] = u * v' + v * u'$. From an anti-derivative point of view:

$$\begin{aligned} \int [u * v' \quad + \quad v * u']\, dx &= u * v \\ \int u\, dv \quad + \quad \int v\, du \quad &= u * v \\ \int u\, dv \quad &= u * v - \int v\, du \end{aligned}$$

giving us integration by parts. We can also perform integration by parts with a definite integral.

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

Memorize this formula. The trick is in choosing u and dv well: we want the resulting integral to be no more (and hopefully less) intimidating than the original.

Example A: Evaluate $\int x e^x\, dx$. *Answer*: $x e^x - e^x + C$

Example B: Evaluate $\int_1^2 x^2 \ln x\, dx$. *Answer*: $\frac{8}{3}\ln 2 - \frac{7}{9}$

Example C: $\int \ln x\, dx$. *Answer*: $x \ln x - x + C$

Hint: There are some WebAssign questions for which this will be a useful technique.

So, how does one choose u and dv wisely? Herbert E. Kasube, in a 1983 article (*American Mathematical Monthly* 90, pp. 210-211) proposes the acronymn LIATE (logarithmic, inverse trig, algebraic, trig, exponential) for choosing u; dv will be “everything else”. In Example A above, there were no logarithms or inverse trig functions, so $u = x$ was chosen as the algebraic function. In examples B and C u was chosen to be the logarithmic function.

Example D: Evaluate $\int x^2 e^x \, dx$. *Answer*: $e^x\left(x^2 - 2x + 2\right) + C$

Example E: Evaluate $\int x^n e^x \, dx$. *Answer*: $x^n e^x - n\int x^{n-1} e^x \, dx$

This type of formula is called a *reduction formula*. In this case it would be applied n successive times. (Textbook practice exercises address a reduction formula involving trig functions.)

Example F: Evaluate $\int e^x \sin x \, dx$. *Answer*: $\frac{1}{2}e^x\left(\sin x - \cos x\right) + C$

Example G: Evaluate $\int 2x\, e^{x^2} \, dx$. *Answer*: $e^{x^2} + C$

Example H. Evaluate $\int x^2 \sin^{-1} x \, dx$. *Answer*: $\frac{1}{3}x^3 \sin^{-1} x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}\left(1-x^2\right)^{3/2} + C$

This one involves not only integration by parts, but also a semi-complicated integration by substitution.