Calculus 141, section 8.2 Trigonometric Integrals
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Recall that all trig functions can be rewritten in terms of sine and cosine, which means that all integrals involving trig functions can be rewritten as integrals involving powers of sine and cosine, or tangent and secant. Along the way we'll need some variations of previously-discovered identities:

1) $\sin^2 x + \cos^2 x = 1 \implies \tan^2 x + 1 = \sec^2 x$,

2) $\sin 2x = 2 \sin x \cos x \implies \sin x \cos x = \dfrac{1}{2} \sin 2x$

3) $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \implies \sin^2 x = \dfrac{1 - \cos 2x}{2}$ and $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

First, we'll take an integral of the form $\int \sin^m x \cos^n x \, dx$ and rewrite it in a more convenient form.

Example A: $\int \sin^4 x \cos^3 x \, dx$. *Answer*: $\dfrac{1}{5} \sin^5 x - \dfrac{1}{7} \sin^7 x + C$

Example B: $\int \sin^5 x \cos^4 x \, dx$. *Answer*: $-\dfrac{1}{5} \cos^5 x + \dfrac{2}{7} \cos^7 x - \dfrac{1}{9} \cos^9 x + C$

Example C: $\int \sin^2 x \cos^4 x \, dx$. *Answer*: $\dfrac{1}{16} x - \dfrac{1}{64} \sin 4x + \dfrac{1}{48} \sin^3 2x + C$

Whew. What a lot of hoops to jump through—Be sure to do them one at a time.
Special note: The text does this same integral as Example 3, and achieves the same result using a different method which you may want to look at. My method is one more step, but only requires remembering the two formulae in (3) above. The text's method requires formula (2) as well as two other integral formulae.

Now, we turn to integrals of the form $\int \tan^m x \sec^n x \, dx$. We'll need to recall some derivatives:

$\dfrac{d}{dx}\tan x = \sec^2 x$ and $\dfrac{d}{dx}\sec x = \sec x \tan x$ (both of which can be derived via the quotient rule).

Example D: $\int \tan^2 x \sec^4 x \, dx$. *Answer*: $\frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

Example E: $\int \tan^3 x \sec^4 x \, dx$ *Answer*: $\frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C$

Note that we could have used the method of Example D, getting $\dfrac{1}{6}\tan^6 x + \dfrac{1}{4}\tan^4 x + C$ which can be shown to be equivalent by substituting identity (1) and simplifying.

Example F: $\int \tan^2 x \, \sec x \, dx$ *Answer*: $\frac{1}{2}\sec x \tan x - \frac{1}{2}\ln\left|\sec x + \tan x\right| + C$

There's one more process you need to be familiar with for the practice exercises and WebAssign: working with integrals of the form $\int \sin(ax)\cos(bx)\,dx$, which are first simplified using an identity you may recall from Precalculus: $\sin(ax)\cos(bx) = \dfrac{1}{2}\sin(a-b)x + \dfrac{1}{2}\sin(a+b)x$.

Example G: $\int \sin(6x)\cos(5x)\,dx$. *Answer*: $-\frac{1}{2}\cos x - \frac{1}{22}\cos(11x) + C$