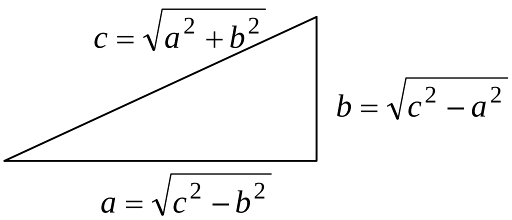
Calculus 141, section 8.3 Trig Substitution

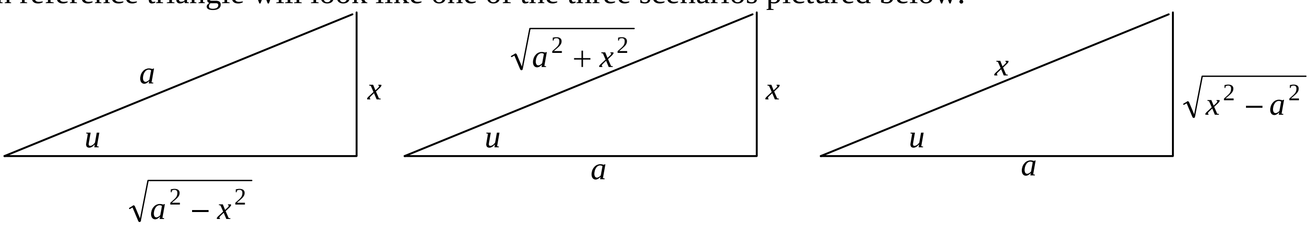
notes by Tim Pilachowski

We've done substitutions up to this point in an effort to transform the complicated into a less complicated form. In this case, we're doing the same thing, although it may temporarily look more complicated before it looks less complicated. The complication in these examples will be the quadratic forms shown in the reference triangles below. We'll substitute a trig function in to get a simpler form, integrate, then back-substitute to regain the original form. Got it?



First of all, recall that the Pythagorean formula allows us to rewrite each side of a right triangle in terms of the other two sides. The version we choose will be determined by the information given to us in a particular problem—we want to choose the version which will give us a substitution in the form *x* = (constant) times (trig function of *u*).

Each reference triangle will look like one of the three scenarios pictured below.



1) $\sin u = \dfrac{x}{a}\left(=\dfrac{opp}{hyp}\right)$ 2) $\tan u = \dfrac{x}{a}\left(=\dfrac{opp}{adj}\right)$ 3) $\sec u = \dfrac{x}{a}\left(=\dfrac{1}{\cos u}=\dfrac{hyp}{adj}\right)$

1) $x = a\sin u \quad \sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 u}=\sqrt{a^2\cos^2 u}=a|\cos u|$

2) $x = a\tan u \quad \sqrt{a^2+x^2}=\sqrt{a^2+a^2\tan^2 u}=\sqrt{a^2\sec^2 u}=a|\sec u|$

3) $x = a\sec u \quad \sqrt{x^2-a^2}=\sqrt{a^2\sec^2 u-a^2}=\sqrt{a^2\tan^2 u}=a|\tan u|$

In each case *u* equals an inverse trig function of (*x*/*a*). Note especially the restriction on *u* which will make the absolute values above unnecessary.

$$x = a\sin u,\quad dx = a\cos u\, du \to u=\sin^{-1}\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2}\le u\le\frac{\pi}{2} \text{ and } \cos u\ge 0$$

$$x = a\tan u,\quad dx = a\sec^2 u\, du \to u=\tan^{-1}\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2}< u<\frac{\pi}{2} \text{ and } \sec u> 0$$

$$x = a\sec u,\quad dx = a\sec u\tan u\, du \to u=\sec^{-1}\left(\frac{x}{a}\right) \text{ with } 0< u<\frac{\pi}{2} \text{ or } \pi< u<\frac{3\pi}{2} \text{ and } \tan u> 0$$

Important note: While the examples below will contain some features similar to the integrals of section 7.5 which evaluated into inverse trig functions, they are not exactly the same. You could, however, use the methods below to evaluate the integrals of 7.5.

Now that the preliminaries are all set, let's get down to business.

Example A: $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$ Note that the domain is $(-3, 3)$. *Answer*: $-9\sqrt{9-x^2}+\frac{(9-x^2)^{3/2}}{3}+C$

Example B: $\int \frac{dx}{\sqrt{25x^2-4}}$ Note that the domain is $\left(\frac{2}{5}, \infty\right)$. *Answer*: $\frac{1}{5}\ln\left|\frac{5x}{2}+\frac{\sqrt{25x^2-4}}{2}\right|+C$

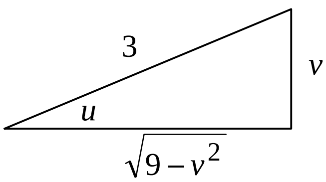
Example C: Find the volume, V, of a solid generated by revolving the function $f(x)=\frac{4}{x^2+4}$ about the x-axis on the interval [0, 2]. *Answer*: $V=\pi\left(\frac{\pi}{4}+\frac{1}{2}\right)$

A second method is to use $\sin 2u = 2\sin u\cos u$, derive $\sin u$ and $\cos u$ from the reference triangle, use back substitution to rewrite V in terms of x, then evaluate from 0 to 2. Since this involves a good bit more effort than the method above, we'll skip it in the name of efficiency—the process is done in detail in Example D below.

Example D: $\int \sqrt{-2x^2+16x-23}\,dx$. *Answer*: $\frac{9\sqrt{2}}{4}\sin^{-1}\left[\frac{\sqrt{2}}{3}(x-4)\right]+\frac{x-4}{2}\sqrt{-2x^2+16x-23}+C$

We can use trig substitution to solve integrals involving quadratics of the form bx^2+cx+d by using completing the square.

$$\int \sqrt{-2x^2+16x-23}\,dx = \int \sqrt{-2\left(x^2-8x\right)-23}\,dx = \int \sqrt{-2\left(x^2-8x+16\right)+2(16)-23}\,dx$$
$$= \int \sqrt{-2(x+4)^2+9}\,dx = \int \sqrt{9-2(x+4)^2}\,dx$$



$$v^2 = 2(x+4)^2,\ \ v=\sqrt{2}\,(x+4),\ \ dv=\sqrt{2}\,dx \ \Rightarrow\ \frac{1}{\sqrt{2}}\int \sqrt{9-v^2}\,dv$$
$$\sin u = \frac{v}{3},\ v=3\sin u,\ dv = 3\cos u\,du$$
$$9-v^2 = 9-9\sin^2 u = 9\left(1-\sin^2 u\right) = 9\cos^2 u$$

$$\frac{1}{\sqrt{2}}\int \sqrt{9-v^2}\,dv = \frac{1}{\sqrt{2}}\int \sqrt{9\cos^2 u}\,(3\cos u)\,du = \frac{9}{\sqrt{2}}\int \cos^2 u\,du = \frac{9}{\sqrt{2}}\int \frac{1+\cos 2u}{2}\,du$$
$$= \frac{9}{2\sqrt{2}}\left(u+\sin 2u\right)+C = \frac{9}{2\sqrt{2}}\left(u+2\sin u\cos u\right)+C$$

Back substitutions from *u* to *v* to *x* get us to a final answer.

$$v = 3\sin u \ \Rightarrow\ u=\sin^{-1}\left(\frac{v}{3}\right)=\sin^{-1}\left[\frac{\sqrt{2}\,(x-4)}{3}\right]$$

Use triangle ratios to find sin *u* and cos *u*.

$$\sin u = \frac{v}{3} = \frac{\sqrt{2}(x+4)}{3},\ \ \cos u = \frac{\sqrt{9-v^2}}{3} = \frac{\sqrt{-2x^2+16x-23}}{3}$$

$$\frac{9}{2\sqrt{2}}u+\frac{9}{2\sqrt{2}}\sin u\cos u + C = \frac{9}{2\sqrt{2}}\sin^{-1}\left[\frac{\sqrt{2}}{3}(x-4)\right]+\left(\frac{9}{2\sqrt{2}}\right)\left(\frac{\sqrt{2}(x-4)}{3}\right)\frac{\sqrt{-2x^2+16x-23}}{3}+C$$
$$= \frac{9\sqrt{2}}{4}\sin^{-1}\left[\frac{\sqrt{2}}{3}(x-4)\right]+\frac{x-4}{2}\sqrt{-2x^2+16x-23}+C$$