Calculus 141, section 8.7a Improper Integrals

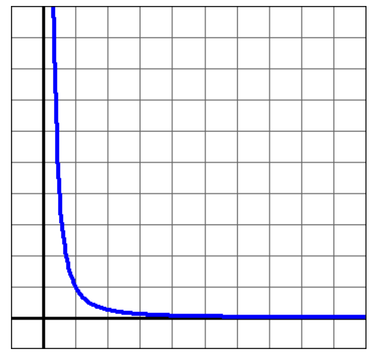
notes by Tim Pilachowski

The definite integrals encountered so far have been *proper*, i.e. they have been evaluated over a finite interval on which the function is continuous. *Improper* integrals take one of two forms: a) the boundaries of integration are *unbounded* (i.e. they go to ∞, – ∞, or both), or b) the integrand is unbounded on the interval of integration.

Think about the limits we've done: some go to a specific finite value, others go to infinity, still others do not exist. We will, in fact, use limits to evaluate improper integrals. The basic method is to rewrite the integral as a limit — $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$, with associated versions for integrals involving $-\infty$ and denominators approaching 0.

Improper integrals that have a numeric value are said to be *convergent*. The rest are said to be *divergent*.

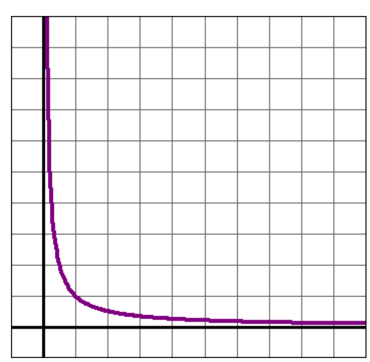
Example A: Does $\int_1^\infty \frac{dx}{x^2}$ converge? *Answer*: converges to 1



Example A extended: Does $\int_0^\infty \frac{dx}{x^2}$ converge? *Answer*: diverges

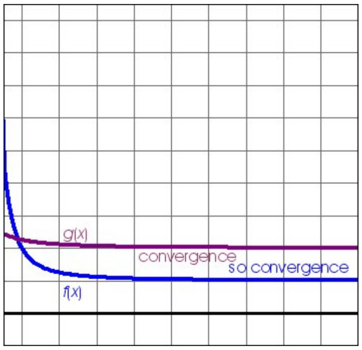
Example B: Does $\int_{-\infty}^\infty \frac{dx}{x^2+1}$ converge? *Answer*: converges to π

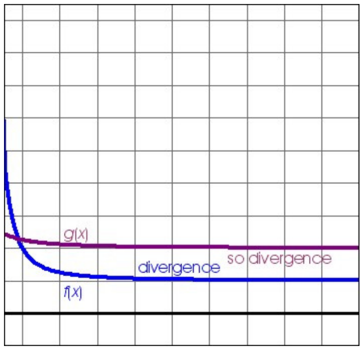
Example C: Does $\int_1^\infty \frac{dx}{x}$ converge? *Answer*: diverges



Comparison did not help us in Example C, but sometimes it can. The Comparison Property (Theorem 8.3) states

For $f(x)$ continuous on the interval $[a,\infty]$ and $0 \le f(x) \le g(x)$ for $a \le x < \infty$,





if $\int_a^\infty g(x)\,dx$ converges, so does $\int_a^\infty f(x)\,dx$ and if $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$.

Example D: Does $\int_1^\infty \frac{dx}{x^3+1}$ converge? *Answer*: converges to a value less than $\frac{\pi}{2}$

Example E: Does $\int_1^\infty \frac{dx}{\sqrt{x}}$ converge? *Answer*: diverges

Example F: Does $\int_1^\infty \cos x\,dx$ converge? *Answer*: diverges

One more thing: the text discusses the *normal probability distribution*—a very useful tool in statistical analysis. You should read through this portion of the book, but are not required to memorize the formulas for this function.