Calculus 141, section 9.1 Taylor polynomial approximation ~ Introduction

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In the previous section, we were able to approximate the value of an integral using first rectangles (midpoint sum), then trapezoids, then quadratics (Simpson's Rule). In chapter 9 we turn to a similar process for approximating any curve, but extend it to using higher degree polynomials and higher-order derivatives.

We'll need the definition of a *factorial* to make the notation of our process a little easier: "n factorial" is written and defined as $n! = n(n-1)(n-2)\ldots(3)(2)(1)$ for integers $n \geq 1$, and $0! = 1$. For example:
$8! = (8)(7)(6)(5)(4)(3)(2)(1) = 40320$ $\quad 5! = (5)(4)(3)(2)(1) = 120$ $\quad 3! = (3)(2)(1) = 6$

Now, recall how the definition of the first derivative was developed, i.e. as a linear expression of the tangent line at a given value of *x*. In slope-intercept form $(y = b + mx)$ for *x* = 0, we'd have $l(x) = f(0) + f'(0)x$.

Can we develop a similar quadratic expression? If so, we'd want it to have the following properties:

$q(x) = c + bx + ax^2$ with $q(0) = c$ such that $q(0) = f(0)$

$q'(x) = b + 2ax$ with $q'(0) = b$ such that $q'(0) = f'(0)$

$q''(x) = 2a$ with $q''(0) = 2a \rightarrow \dfrac{q''(0)}{2} = a$ such that $q''(0) = f''(0)$

Thus, our quadratic approximation would be $q(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2$. (See section 3.8 of the text.)

Let's get bold and daring, and go for a fifth-degree approximation, with the following properties:

$p_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ with $p_5(0) = a_0$ such that $p_5(0) = f(0)$

$p_5^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4$ with $p_5^{(1)}(0) = a_1$ such that $p_5^{(1)}(0) = f^{(1)}(0)$

$p_5^{(2)}(x) = 2a_2 + 3*2a_3 x + 4*3a_4 x^2 + 5*4a_5 x^3$ with $\dfrac{p_5^{(2)}(0)}{2} = a_2$ such that $p_5^{(2)}(0) = f^{(2)}(0)$

$p_5^{(3)}(x) = 3*2a_3 + 4*3*2a_4 x + 5*4*3a_5 x^2$ with $\dfrac{p_5^{(3)}(0)}{3*2} = a_3$ such that $p_5^{(3)}(0) = f^{(3)}(0)$

$p_5^{(4)}(x) = 4*3*2a_4 + 5*4*3*2a_5 x$ with $\dfrac{p_5^{(4)}(0)}{4*3*2} = a_4$ such that $p_5^{(4)}(0) = f^{(4)}(0)$

$p_5^{(5)}(x) = 5*4*3*2a_5$ with $\dfrac{p_5^{(5)}(0)}{5*4*3*2} = a_5$ such that $p_5^{(5)}(0) = f^{(5)}(0)$

Thus, our fifth-degree approximation is

$p_5(x) = f(0) + f^{(1)}(0)x + \dfrac{f^{(2)}(0)}{2!}x^2 + \dfrac{f^{(3)}(0)}{3!}x^3 + \dfrac{f^{(4)}(0)}{4!}x^4 + \dfrac{f^{(5)}(0)}{5!}x^5$.

If we continue this process for higher-order polynomials, we would get the *n*th Taylor polynomial of *f* about 0:

$$p_n(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

(When we get to section 9.9, we'll develop a more general formula for a Taylor expansion about values other than 0, and include a formula for estimating the error involved in such an approximation.)

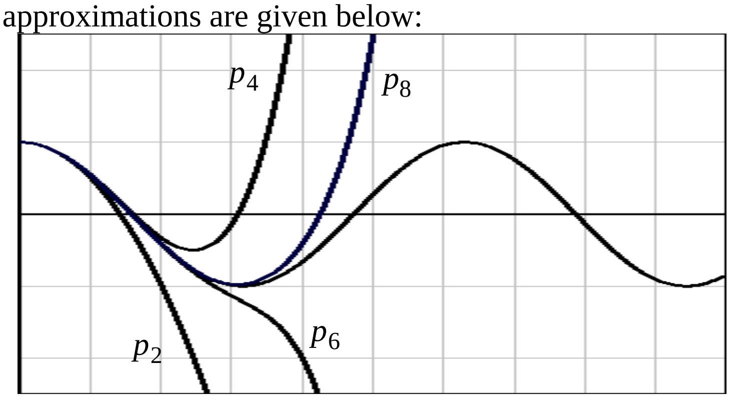
Example A: Given a function *f* such that $f(0) = 1$, $f^{(1)}(0) = -3$, $f^{(2)}(0) = 5$, $f^{(3)}(0) = -7$ find the third-degree Taylor polynomial approximation. *Answer*: $p_3(x) = 1 - 3x + \frac{5}{2}x^2 - \frac{7}{6}x^3$

The text develops the Taylor approximations for $f(x)=e^x$ for n = 10 and $f(x)=\ln(1+x)$ for n = 10 and n = 20 in a very straightforward manner, so we won't recreate those. Instead, let's consider a trigonometric function.

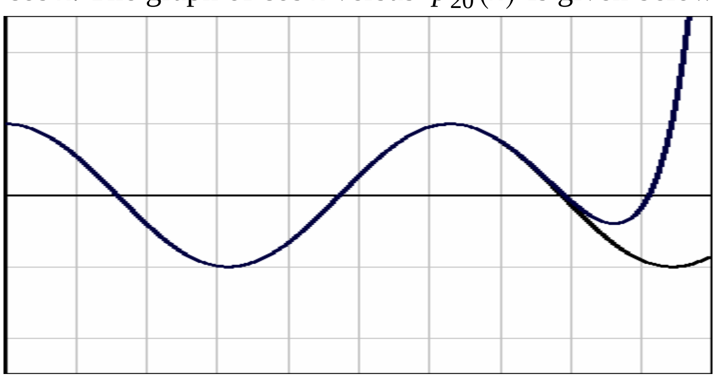
Example B: Find $p_2(x)$, $p_3(x)$, $p_4(x)$, $p_6(x)$, and $p_8(x)$ for $f(x)=\cos x$.

Answers: $1-\frac{1}{2}x^2$; $1-\frac{1}{2}x^2$; $1-\frac{1}{2}x^2+\frac{1}{24}x^4$; $1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$; $1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\frac{1}{40320}x^8$

Graphs of *f* and our Taylor approximations are given below:



Note that the polynomials each veer away from cos *x* as *x* gets larger, but that the higher-degree approximations "follow" cos *x* longer than the lower-degree polynomials, i.e. as the oscillations of the polynomial more closely match the periodic nature of cos *x*. The graph of cos *x* versus $p_{20}(x)$ is given below.



In later topics we will address questions of whether our approximations converge or diverge for given values of *x*, how we can know the difference, and (in cases where the approximations converge) the degree of the approximation needed to contain errors within a given margin. In particular for cos *x*, we'll be able to show that the Taylor approximations as derived above *do* converge to cos *x* as *n* approaches ∞.