Calculus 141, section 9.2 Sequences

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A sequence is a list of numbers, given in a particular order, found according to a particular formula—in other words, a function. The domain of a sequence consists of non-negative integers, with the particular terms of a sequence designated as $a_0, a_1, a_{3,}$ etc., and the general term of a sequence most often designated as a_n. Each subscript is called an index.

Example A: The sequence defined by $a_n = \frac{n}{n+1}$ for n = 1, 2, 3, … gives us terms $a_1 = \frac{1}{2}$, $a_2 = \frac{2}{3}$, $a_3 = \frac{3}{4}$, etc. This sequence can be defined with the notation $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$.

A couple of sequences are useful enough to have their own names. Practice problems 51 and 53 from the text have you investigate compound interest, done as both compounding n times per year and as continuous compounding. The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is called the *harmonic* sequence. The sequence $\left\{r^n\right\}$is a *geometric* sequence, which the text shows converges to 0 for $|r| < 1$, converges to 1 for r = 1, and diverges for all other values of r (Example 7).

But what does it mean to say that a sequence converges, i.e. whether or not a particular sequence has a limit equal to a particular numeric value? If we plot the terms of $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ as points on a graph, the sequence seems to converge to a value of 1, but as always, "seems to" is insufficient for mathematical purposes.

Definitions: A number L is the limit of a sequence if for every $\varepsilon > 0$ there is an integer N such that if $n \geq N$, then $|a_n - L| < \varepsilon$, and we'll say that the sequence *converges* to L. If, on the other hand, for every number M there is an integer N such that if $n \geq N$, then $a_n > M$, we'll say that the sequence *diverges* to ∞. If $a_n < M$ we'll say that the sequence diverges to $-\infty$.

Example A continued: Show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Note that we have proven that the sequences $\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty}$, $\left\{\frac{n}{n+1}\right\}_{n=10}^{\infty}$, and $\left\{\frac{n}{n+1}\right\}_{n=100}^{\infty}$ also converge! The behavior of the sequence at its beginning is not the determining criterion. Rather it is the behavior as n approaches infinity which concerns us.

Example B: Determine whether the sequence $\{n!\}_{n=0}^{\infty}$ converges or diverges.

Fortunately, we don't have to rely on the definitions above to show convergence and divergence. Recall that a sequence is a function, defined on a domain consisting of non-negative integers. It is not difficult to show that if the related function *f*(*x*), defined over a domain consisting of all positive numbers, has a limit, then the sequence, defined over a more limited part of that domain, also has a limit (Theorem 9.4).

Example C: Find $\lim_{n\to\infty} \frac{\sin n}{n}$.

Example D: Does the sequence $\{\sin n\}_{n=0}^{\infty}$ converge?

Example A one more time: If we let $f(x)=\frac{x}{x+1}$ for $x \geq 1$, then since $\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1} = 1$ (L'Hôpital's Rule) we can say that the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ converges to 1.

Example E: Does the sequence $\left\{\frac{2^n}{n^2}\right\}_{n=1}^{\infty}$ converge?

Example F: Does the sequence $\left\{n^{1/n}\right\}_{n=1}^{\infty}$ converge?