

Calculus 141, section 9.3 Convergence Properties of Sequences

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Because a sequence is a function, the properties of limits as they apply to functions also apply to sequences:

$$\lim_{n \rightarrow \infty} a_n + b_n = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n \quad \lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n \quad \lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} b_n \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$$

These allow us to evaluate some otherwise intimidating sequences.

Example A: Determine whether or not the sequence $\left\{ \sqrt{n^2 + 3n} - n \right\}_{n=1}^{\infty}$ converges. (This is a bit more complicated than ones you'll be asked to do in practice and in WebAssign, but the simplifying process contains within it a number of useful strategies.) *Answer:* converges to $\frac{3}{2}$

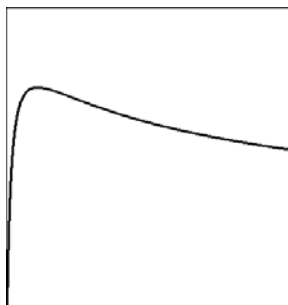
We can use the Squeezing Theorem also: If $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$, and there is a sequence $\{c_n\}_{n=m}^{\infty}$ such that

$a_n \leq c_n \leq b_n$ for $n \geq m$, then $\{c_n\}_{n=m}^{\infty}$ converges and $\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.

Example B: Determine whether or not $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$ converges. *Answer:* converges to 0

Side note: In the Lecture notes for 9.2, we showed that $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$. This result can be verified using the Squeezing Theorem.

Theorem 9.5 from the text shows that if $\{a_n\}_{n=m}^{\infty}$ converges, then $\{a_n\}_{n=m}^{\infty}$ is bounded. The contrapositive, which is logically equivalent says that if $\{a_n\}_{n=m}^{\infty}$ is unbounded then $\{a_n\}_{n=m}^{\infty}$ diverges. The converse, however, is *not* logically equivalent. That is, if a sequence is bounded it may or may not converge. For example, $\{(-1)^n\}$ and $\{\sin n\}$ are both bounded, but they oscillate rather than converge. An additional requirement is necessary (Theorem 9.6): A bounded sequence that is either increasing or decreasing converges.



Example C: Determine whether or not $\left\{ \frac{\ln n}{\sqrt{n}} \right\}_{n=1}^{\infty}$ converges. *Answer:* It does.

You do not need to know the following for Math 141, but may find the idea useful at a later time.

A sequence can be defined recursively, with a basis that provides values for the first term (or terms) and a *recursion* that defines succeeding terms with a formula that involves preceding terms.

Example D: Compound interest can be described recursively as follows, with

a_n = balance after the n th interest payment is added

r = annual interest rate

t = the time period for which interest is calculated (quarterly $t = \frac{1}{4}$, monthly $t = \frac{1}{12}$)

balance in the account = $\begin{cases} a_0 = \text{initial deposit} \\ a_n = a_{n-1}(1 + rt) \end{cases}$.

The text notes that the Newton-Raphson method of section 3.8 is defined recursively.

It also uses the famous Fibonacci sequence in the Project for this section. The Fibonacci sequence defines the recursion in terms of the preceding *two* terms—you might find it interesting.