Calculus 141, section 9.4 Infinite Series

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A ***sequence*** $\{a_n\}_{n=m}^{\infty}$ consists of an ordered set of numbers. If we were to begin adding the numbers of a sequence together, $s_j = a_m + a_{m+1} + \ldots + a_{m+j-1}$, we would have a ***partial sum***, designated as $\sum_{n=m}^{m+j-1} a_n$. The sum s_j above of the first *j* terms is called the *j*th partial sum. If we take the sum to infinity, $\sum_{n=m}^{\infty} a_n$, then we have an ***series***, that is, an infinite sum. But what does it mean to add an infinite sequence of numbers, and can such an infinite series add up to a finite number? In other words, does an infinite series have a limit, does it converge?

Definition: Given a sequence $\{a_n\}_{n=m}^{\infty}$ and a series $\sum_{n=m}^{\infty} a_n$, if $\lim_{j\to\infty}\left(a_m + a_{m+1} + \ldots + a_{m+j-1}\right)$ exists, then $\sum_{n=m}^{\infty} a_n = \lim_{j\to\infty}\left(a_m + a_{m+1} + \ldots + a_{m+j-1}\right)$ and the series converges. If the limit goes to ∞ or does not exist, then the series diverges.

How can we determine whether or not a series converges? One way to approach this is to remember that we have ***two*** sequences involved in every series. The first is the sequence of terms $\{a_n\}_{n=m}^{\infty}$. The second is the sequence of partial sums: $s_m = a_m$, $s_{m+1} = a_m + a_{m+1}$, $s_{m+2} = a_m + a_{m+1} + a_{m+2}$, $\ldots$. If we can show that the sequence of partial sums $\{s_j\}$ converges, we will be able to conclude that the series converges.

Example A: Does the series $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converge, and if so, to what limit?

Intuitive Answer: converges to what value?

Be careful that you are clear in your own mind about the differences among the sequence, the sequence of partial sums, and the series.

sequence $\{a_n\} = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$ $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

sequence of partial sums $\{s_j\} = \left\{2 - \frac{1}{2^j}\right\}_{j=0}^{\infty}$ $2-1,\ 2-\frac{1}{2},\ 2-\frac{1}{4},\ 2-\frac{1}{8},\ 2-\frac{1}{16}, \ldots$

series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

Example A extended: Find $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=2}^{\infty} \frac{1}{2^n}$. *Answers*: converges to 1; converges to $\frac{1}{2}$

Example A extended again: Find $\sum_{m=0}^{\infty} \frac{1}{2^{m+1}}$ and $\sum_{m=0}^{\infty} \frac{1}{2^{m+2}}$. *Answers*: converges to 1; to $\frac{1}{2}$

Example B: Determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{n^2+7n+12}$ converges and if so, find its limit.

Answer: converges to $\frac{1}{4}$

Example C: Determine whether $\sum_{n=1}^{\infty} (-1)^n$ converges, and if so, find its limit. *Answer*: diverges

The text proves a couple of theorems that formalize the relationship between a series and its related sequence. Theorem 9.8 states: If $\sum_{n=m}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. The contrapositive, which is logically equivalent is stated as Corollary 9.9: If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=m}^{\infty} a_n$ diverges.

Note that in Examples A and B above (convergent series), $\lim_{n\to\infty} \frac{1}{2^n} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2 + 7n + 12} = 0$, while in Example C (divergent series), $\lim_{n\to\infty} (-1)^n$ does not exist. Also, from Lecture 9.3 Example A in, we now know that $\sum_{n=1}^{\infty} \sqrt{n^2 + 3n} - n$ diverges since $\lim_{n\to\infty} \sqrt{n^2 + 3n} - n = \frac{3}{2} \neq 0$.

Important cautionary note (again): Be sure that you are clear in your own mind that there is a difference between the sequence of terms for $\{a_n\}_{n=m}^{\infty}$ and the sequence of partial sums $\{s_j\}$ produced by the series $\sum_{n=m}^{\infty} a_n$.

The converse of Theorem 9.8 is ***not*** true: $\lim_{n\to\infty} a_n = 0$ is not a guarantee that $\sum_{n=1}^{\infty} a_n$ will converge.

Although the harmonic sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 0, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (see text Example 4). Thus (Lecture 9.3 Example B) since the sequence $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$ converges to 0, although we can say that the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ ***might*** converge, we cannot be certain that it ***does*** converge.

The text also proves some very convenient theorems which ease the task of evaluating series. Theorem 9.10 states: For $c \neq 0$ and $m \geq 0$, the geometric series $\sum_{n=m}^{\infty} c\,r^n$ converges if and only if $|r| < 1$, and if it converges we can calculate the sum of the series: $\sum_{n=m}^{\infty} c\,r^n = \frac{c\,r^m}{1-r}$.

The proof relies on calculating the sequence of partial sums and identifying the pattern which emerges.

Example A, $\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $c = 1$ and $r = \frac{1}{2}$, and

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1\left(\frac{1}{2}\right)^0}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{2^n} = \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

Theorem 9.11 and associated theorems give us means of combining series in much the same way that we can add (and subtract) and multiply (and divide) limits:

a. If $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ both converge, then $\sum_{n=m}^{\infty} (a_n + b_n)$ converges and $\sum_{n=m}^{\infty} (a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n$.

b. For any number c, if $\sum_{n=m}^{\infty} a_n$ converges, then $\sum_{n=m}^{\infty} c\,a_n$ also converges and $\sum_{n=m}^{\infty} c\,a_n = c * \sum_{n=m}^{\infty} a_n$.

Example D: Evaluate $\sum_{n=0}^{\infty} \frac{4+2^n}{3^{n+1}}$. *Answer*: converges to 3

Method: Use factoring and separation of fractions to rearrange the series into the form of a geometric series and use the formula from Theorem 9.10 to evaluate: $\sum_{n=m}^{\infty} c\,r^n = \frac{c\,r^m}{1-r}$ with $|\,r\,| < 1$.

factor: $$\sum_{n=0}^{\infty} \frac{4+2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{4+2^n}{3^n}$$

separate the fractions (distribute the division): $$= \frac{1}{3}\sum_{n=0}^{\infty} \frac{4}{3^n} + \frac{1}{3}\sum_{n=0}^{\infty} \frac{2^n}{3^n}$$

rewrite as $c\,r^n$: $$= \frac{4}{3}\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$$

apply Theorem 9.10: $$= \frac{\frac{4}{3}\left(\frac{1}{3}\right)^0}{1-\frac{1}{3}} + \frac{\frac{1}{3}\left(\frac{2}{3}\right)^0}{1-\frac{2}{3}}$$

simplify: $$= \frac{\frac{4}{3}}{\frac{2}{3}} + \frac{\frac{1}{3}}{\frac{1}{3}} = 2+1 = 3$$