Calculus 141, section 9.5 Integral Test and Comparison Tests

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Consider series such as $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$. You may notice that these resemble improper integrals $\int_1^{\infty} \frac{dx}{x}$, $\int_1^{\infty} \frac{dx}{\sqrt{x}}$, $\int_1^{\infty} \frac{dx}{x^2}$, and $\int_1^{\infty} \frac{dx}{2^x}$. Indeed, both $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series, section 9.4) and $\int_1^{\infty} \frac{dx}{x}$ (section 8.7) diverge. Might we suspect a similar relationship between the other infinite series and corresponding improper integrals?

By observing the relationships between the terms a_n of a positive decreasing sequence, and the values of the function f for which $f(n) = a_n$, the text proves Theorem 9.12, the Integral Test: For a positive decreasing sequence $\{a_n\}_{n=1}^{\infty}$, and f a continuous function on $[1, \infty)$ such that $f(n) = a_n$, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges.

Examples A: Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$, and $\int_1^{\infty} \frac{dx}{2^x}$ converge or diverge.

Answers: diverges, converges, converges

Example B: Determine values of p for which $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges and diverges. *Answers*: $p > 1$; $0 < p \leq 1$

Example C: Does $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converge? *Answer*: yes

Example D: Does the infinite series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converge? *Answer*: no

Example A revisited: Determine the value to which $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. *Answer*: $\frac{\pi^2}{6}$ (trick question)

The question now becomes: How quickly and how closely do the partial sums generated by a series approach the actual value of the series? The *j*th truncation error is defined to be

$E_j = \left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{j} a_n \right| = \left| \sum_{n=j+1}^{\infty} a_n \right|$, i.e. the rest of the sequence beyond the *j*th partial sum. The text applies the Integral Test to show that $\int_{j+1}^{\infty} f(x)\,dx \le E_j \le \int_{j}^{\infty} f(x)\,dx$. In Example 2 the text estimates E_{100} for $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Example C revisited: Estimate E_{10000} for the series $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$. *Answer*: within 0.02

Recall section 8.7, where we developed the comparison test for improper integrals. A similar test exists for infinite sums: Theorem 9.13 gives us the Direct Comparison Test for infinite sums.

a. If $\sum_{n=1}^{\infty} b_n$ converges and $0 < a_n \le b_n$ for all $n \ge 1$, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$.

b. If $\sum_{n=1}^{\infty} b_n$ diverges and $0 < b_n \le a_n$ for all $n \ge 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example E: Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge? *Answer*: yes

Direct comparison is sometimes awkward, or at least inconvenient. The Limit Comparison Test provides an alternative (Theorem 9.14): If $\lim_{n \to \infty} \frac{a_n}{b_n}$ exists and is a positive number, then positive series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. To apply the Limit Comparison Test, we'll look for a sequence $\{b_n\}$ which has known properties and whose *n*th term behaves in a fashion similar to the *n*th term of the sequence $\{a_n\}$ for large values of *n*. If we can determine that $\lim_{n \to \infty} \frac{a_n}{b_n}$ is a positive number, we'll be able to draw a conclusion about the series $\sum_{n=1}^{\infty} a_n$.

Example F: Does $\sum_{n=1}^{\infty} \frac{1}{2^n - 5}$ converge? *Answer*: yes

Example G: Does $\sum_{n=1}^{\infty} \frac{3n+2}{\sqrt{n}(3n-5)}$ converge? *Answer*: no