Calculus 141, section 9.6 Ratio Test and Root Tests

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- The geometric series $\sum_{n=m}^{\infty} c r^n = \frac{c r^m}{1-r}$ if and only if $|r| < 1$.
- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges whenever $p > 1$ and diverges whenever $0 < p \le 1$.
- The Integral Test states a series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.
- In the Direct Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges if its terms are less than those of a known convergent series, and diverges if its terms are greater than those of a known convergent series.
- The Limit Comparison Test states: If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is a positive number, then positive series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

A downside to the Comparison Tests is that they require a suitable series to use for the comparison.
In contrast, the Ratio Test and the Root Test require only the series itself.

Ratio Test (Theorem 9.15) Given a positive series $\sum_{n=1}^{\infty} a_n$ for which $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$: a. If $0 \le r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. b. If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. c. If $r = 1$ or $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist, no conclusion.

The proof of part a. relies upon the definition of limits and the "creation" of a geometric series which converges to which we compare our original series. Briefly, by the nature of inequalities there exists a value s for which $0 \le r < s < 1$. From the definition of limits, there exists a value N such that for $n \ge N$, $\frac{a_{n+1}}{a_n} \le s \;\Rightarrow\; a_{n+1} \le a_n s$.

Then $0 < a_{N+2} \le a_{N+1} s \le (a_N s)s = a_N s^2 \;\Rightarrow\; 0 < a_{N+n} \le a_N s^n$. The latter value is the basis for a geometric series which converges. The application of the Comparison Test finishes the proof of part a. The proof of part b. is analogous, with $r > s > 1$.

Hint: The Ratio Test works best for series such as $\sum \frac{1}{n!}$, $\sum r^n$, and $\sum \frac{1}{2^n + c}$ for which n is a factorial or an exponent.

Example A: Does $\sum_{n=0}^{\infty} \frac{100^n}{n!}$ converge? *Answer*: yes

Example B: Does $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converge? *Answer*: no

Example B upside down: Does the "reciprocal" series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge? *Answer*: yes

In Lecture 9.4 it was noted that since $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$ converges to 0, although we could say that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ *might* converge, we could not be certain that it *does* converge.

The nature of the Ratio Test is such that if it shows a series converges, then the series involving the reciprocals of the terms must diverge, and vice-versa.

Example C: Use the Ratio Test to test convergence of $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Answer*: inconclusive

Root Test (Theorem 9.16) Given a positive series $\sum_{n=1}^{\infty} a_n$ for which $\lim_{n\to\infty} \sqrt[n]{a_n} = r$: a. If $0 \le r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. b. If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. c. If $r = 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = r$ DNE, no conclusion.

The proof is a little simpler than that for the Ratio Test: For values *s* and *N* as described above, $\sqrt[n]{a_n} \le s \Rightarrow a_n \le s^n$, the basis for a geometric series which converges. The Comparison Test finishes the proof of part a. The proof of part b. is analogous, with $r > s > 1$. The Root Test is especially useful in series that involve a *k*th power and which have no complications such as factorials.

Example D: Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converge? *Answer*: yes

Example E: Does $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$ converge? *Answer*: no