Calculus 141, section 9.7 Alternating Series, Absolute Convergence

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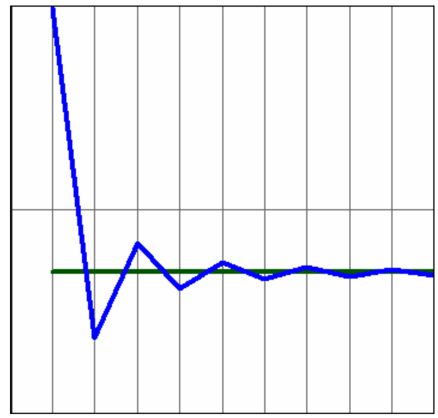
So far, we have pretty much limited our attention to series which are positive. What can we say of those which are not positive? We have taken a quick look at one alternating series: $\sum_{n=1}^{\infty}(-1)^n$ diverges (Example C in lecture notes 9.4) because the sequence of partial sums does not converge to a single value, but rather alternates between -1 and 0. The first question becomes: Can we determine whether an alternating series is convergent or divergent? Theorem 9.17, credited to Leibniz, provides a straightforward test. To show that the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges, one need only to show that the sequence $\{a_n\}_{n=1}^{\infty}$ is positive and decreasing, and that $\lim_{n\to\infty} a_n = 0$. Because the terms alternate in sign, the partial sums successively rise and fall. Because the sequence $\{a_n\}_{n=1}^{\infty}$ is positive and decreasing, the sequence of partial sums will alternately "overshoot" and "undershoot" the limiting value, in a manner similar to the function pictured to the right.



Example A: Does the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\ldots$ converge? *Answer*: yes

Example B: Does the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1-\frac{1}{4}+\frac{1}{9}-\ldots$ converge? *Answer*: yes

Example C: Does the alternating series $\sum_{n=1}^{\infty}(-1)^{n}\frac{2n+1}{3n-1} = -\frac{3}{2}+1-\frac{7}{8}+\frac{9}{11}-\ldots$ converge? *Answer*: no

Example D: Does the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{2n-1}=1-\frac{\sqrt{2}}{3}+\frac{\sqrt{3}}{5}-\ldots$ converge? *Answer*: yes

Example E: Does the alternating series $\sum_{n=1}^{\infty}(-1)^{n}\left(\frac{n-1}{n}\right)^{n}=0+\left(\frac{1}{2}\right)^{2}-\left(\frac{2}{3}\right)^{3}+\left(\frac{3}{4}\right)^{4}\ldots$ converge? *Answer*: no

Theorem 9.17 provides a means of estimating the error involved in using a partial sum to approximate a series: the *j*th truncation error satisfies $E_j < a_{j+1}$. The proof relies on the alternating nature of the series. Since the sequence of terms is decreasing, and the partial sums alternately overshoot and undershoot the limit, the error continually decreases by no more than the amount of the succeeding term.

Example A extended: If we use the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ to approximate ln 2, which partial sum would we need to use to be within 10^{-4}? *Answer*: 10,000th partial sum

Example B extended: Which partial sum would we need to approximate $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{2}}$ to within 10^{-4}?

Answer: 100th partial sum

For a convergent series $\sum_{n=1}^{\infty} a_n$, if $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Otherwise, it converges conditionally.

Example A again: Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converge absolutely? *Answer*: no, conditionally

Example B again: Does $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ converge absolutely? *Answer*: yes

We can generalize about an alternating *p*-series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \dots$, which will converge absolutely for all values of $p > 1$ and diverge for $0 < p \le 1$.

Absolute convergence of a series carries with it a benefit useful in evaluating a series which is neither positive nor alternating. By Theorem 9.18, if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Example F: Does $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converge? *Answer*: yes

Theorem 9.20 provides generalized versions of the Direct Comparison Test, Limit Comparison Test, Ratio Test and Root Test. Corollary 9.21 states that if either $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r < 1$ then $\lim_{n\to\infty} a_n = 0$. Likewise, if $r > 1$, then $\lim_{n\to\infty} a_n = \infty$.

N.B. The text provides a summary of convergence tests for series at the end of chapter 9, just before the review pages. I have also put a link on the Math 141 webpage to a similar summary which also includes strategies for choosing which test to use in particular circumstances.

Example G: Find values for x for which $\sum_{n=1}^{\infty} n\,x^n$ converges. *Answer*: $|x| < 1$