

Calculus 141, section 9.8a Power Series & Radius of Convergence

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A series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$ is called a power series. We'll be most interested in the type for which the coefficients form a sequence, and will ask the question: Does this power series converge?

Example ** [Lecture 9.7 Example G]: Find values of x for which $\sum_{n=1}^{\infty} n x^n = x + 2x^2 + \dots$ converges.

Answer: $|x| < 1$

Example A: Determine conditions under which $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ converges. Answer: $|x| < 1$

Lemma 9.23 gives us conditions for convergence of a power series: For a nonzero real number s ,

- If $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = s$, then it converges absolutely for $|x| < |s|$.
- If $\sum_{n=0}^{\infty} c_n x^n$ diverges for $x = s$, then it diverges for $|x| > |s|$.

The “if” in part a., is a big one. We have to determine whether or not the power series converges. But notice that no matter what else might be true, a power series $\sum_{n=0}^{\infty} c_n x^n$ will converge for $x = 0$. At the other extreme, it

is possible to construct a series such as $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges for all values of x (text Example 2).

Examples ** and A above converge within a specific interval. In other words, we get to Theorem 9.24. For any power series, exactly one of the following will apply:

- $\sum_{n=0}^{\infty} c_n x^n$ converges only for $x = 0$, radius of convergence = 0, interval of convergence = $[0, 0]$.
- $\sum_{n=0}^{\infty} c_n x^n$ converges for all values of x , radius of convergence = ∞ , interval of convergence = $(-\infty, \infty)$.
- $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, radius of convergence = $R > 0$, interval of convergence will be $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$. See table 9.1 in the text for examples of each.

Example B: Find the interval of convergence of $\sum_{n=0}^{\infty} 2^n x^n$. *Answer:* $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Example C: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$. *Answer:* $\left[-\frac{1}{2}, \frac{1}{2}\right)$

Example D: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$. *Answer:* $(-\infty, \infty)$

Example E: Find the radius of convergence of $\sum_{n=0}^{\infty} \left(\frac{n+1}{n}\right)^{n^2} x^n$. *Answer:* $R = \frac{1}{e}$

Since we can think of power series as “infinite polynomials” with a domain equaling the interval of convergence, we might ask whether they can be differentiated and integrated. Since derivatives and integrals can be done term by term, we have the following theorems:

Theorem 9.25: For power series $\sum_{n=0}^{\infty} c_n x^n$ with radius of convergence R , the derivative has the same radius of

convergence, and $\frac{d}{dx} \left(\sum_{n=0}^{\infty} c_n x^n \right) = \sum_{n=1}^{\infty} \frac{d}{dx} (c_n x^n) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ for $|x| < R$.

Theorem 9.28: For power series $\sum_{n=0}^{\infty} c_n x^n$ with radius of convergence R , the integral has the same radius of

convergence, and $\int_0^x \left(\sum_{n=0}^{\infty} c_n t^n \right) dt = \sum_{n=0}^{\infty} \left(\int_0^x c_n t^n dt \right) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$ for $|x| < R$. (We’ll use this theorem,

in particular, next time, when we continue section 9.8.).

Example C revisited: Differentiate and integrate $\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1} = 1 + x + \frac{4}{3}x^2 + 2x^3 + \dots$

Example F: Show that the series, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, is its own derivative and integral.

$\frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n \right) = \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = 1 + x + \frac{x^2}{2!} + \dots$ which is the derivative found term by term.

alternate method: by shift of index $\sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = \sum_{(n-1)=1}^{\infty} \frac{1}{((n-1)-1)!} x^{(n-1)-1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$\int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{1}{n!} t^n \right) dt = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{n+1} = (C_0) + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = (C_1) + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

alternate (very non-rigorous): Just as the derivative version changes index from $n = 0$ to $n = 1$, we can accommodate the need for a constant of integration by changing the integration index from $n = 0$ to $n = -1$. But since a negative index is problematic for our use of series, after the integration we’ll shift back to $n = 0$.

$\int \sum_{n=0}^{\infty} \frac{1}{n!} x^n dx = C + \sum_{n=-1}^{\infty} \frac{1}{(n+1)!} x^{n+1} = C + \sum_{(n-1)=-1}^{\infty} \frac{1}{((n-1)+1)!} x^{(n-1)+1} = C + \sum_{n=0}^{\infty} \frac{1}{n!} x^n$