Calculus 141, section 9.8b Playing with Power Series

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For a power series $f(x)=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \ldots$ with radius of convergence = $R > 0$,

$$f'(x)=\sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 \ldots$$

$$f^{(2)}(x)=\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 + 3*2c_3 x + 4*3c_4 x^2 + 5*4c_5 x^3 + 6*5c_6 x^4 \ldots$$

$$f^{(3)}(x)=\sum_{n=3}^{\infty} n(n-1)(n-2) c_n x^{n-3} = 3*2c_3 + 4*3*2c_4 x + 5*4*3c_5 x^2 + 6*5*4c_6 x^3 + 7*6*5c_7 x^4 \ldots$$

$$f^{(4)}(x)=\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) c_n x^{n-4} = 4!c_4 + 5!c_5 x + 6*5*4*3c_6 x^2 + 7*6*5*4c_7 x^3 \ldots$$

Evaluated at $x = 0$ we get $f(0)=c_0$, $f'(0)=c_1$, $f^{(2)}(0)=2!c_2$, $f^{(3)}(0)=3!c_3$, $f^{(4)}(0)=4!c_4$, and in general $f^{(n)}(0)=n!c_n \Rightarrow c_n = \frac{f^{(n)}(0)}{n!}$. Thus we can write $f(x)=\sum_{n=0}^{\infty} c_n x^n = f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for $|x| < R$.

This should look familiar to you as the Taylor expansion about $x = 0$, but with an important difference. In section 9.1 we knew that the Taylor polynomial as valid close to 0, but we had no way to be sure how close we had to be. Now we know that the Taylor polynomial expansion about $x = 0$ is valid as long as we are within the interval of convergence (Theorem 9.26).

This conclusion leads to Corollary 9.27: If $f(x)=\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} b_n x^n$, then $c_n = b_n$ for $|x| < R$. This is sometimes referred to as a uniqueness theorem, and holds true within the interval of convergence. It, along with the derivative and integral theorems, allows us to derive power series expansions for all kinds of functions. A very useful power series to know is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and the text does several derivations using this one. We'll work with $f(x)=\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ (Example A from lecture notes 9.8a) with $R = 1$. The purpose is to illustrate methods that you can use in working with functions expressed as power series.

Example A: Find the power series expansion for $f(x)=\frac{1}{(1-x)^2}$. *Answer*: $\sum_{n=1}^{\infty} n x^{n-1}$

Example A extended: Find the power series expansion for $f(x)=\frac{1}{(1-x)^3}$. *Answer*: $\sum_{n=2}^{\infty}\frac{n(n-1)x^{n-2}}{2}$

Example B: Find the power series expansion for $f(x) = \ln(1 - x)$. *Answer*: $\sum_{n=0}^{\infty}(-1)\left(\frac{1}{n+1}\right)x^{n+1}$

Example C: Find the power series expansion for $f(x) = \ln(1 - x) + x$. *Answer*: $\sum_{n=1}^{\infty}\left(\frac{-1}{n+1}\right)x^{n+1}$

Example D: Find the power series expansion for $f(x)=\frac{\ln(1-x)+x}{x^2}$. *Answer*: $\sum_{n=1}^{\infty}\left(\frac{-1}{n+1}\right)x^{n-1}$

Example E, part 1: Given $f(x)=\frac{\ln(1-x)+x}{x^2}$, find a sensible value to define as $f(0)$. *Answer*: $-\frac{1}{2}$

Example E, part 2: Given $f(x)=\dfrac{\ln(1-x)+x}{x^2}$, find a sensible value to define as $f(1)$. *Answer*: not possible

Example F: Find $\int \dfrac{\ln(1-x)+x}{x^2}\,dx$. *Answer*: $\sum_{n=1}^{\infty}\left(\dfrac{-1}{n\,(n+1)}\right)x^n$

Granted, there may be easier methods for the examples above, but hopefully you have an idea of some things you can do with series. The methods illustrated above have historically been used to find ways to express and evaluate functions and quantities that were otherwise inaccessible. Here's a famous one also developed in the text.

Example G: Approximate $\dfrac{\pi}{4}$ by evaluating $\tan^{-1}(1)$ using a power series expansion. The resulting series can be used to approximate π.

Start with $f(x)=\sum_{n=0}^{\infty}x^n = 1+x+x^2+x^3+\ldots = \dfrac{1}{1-x}$.

Substitute $-x$ for x: $g(x)=f(-x)=\dfrac{1}{1+x}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n\,x^n$

Substitute x^2 for x: $h(x)=g(x^2)=\dfrac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n\,(x^2)^n=\sum_{n=0}^{\infty}(-1)^n\,x^{2n}$

Integrate the series: $\tan^{-1}x=\int\dfrac{1}{1+x^2}\,dx=\int_0^{\infty}\left(\sum_{n=0}^{\infty}(-1)^n\,t^{2n}\right)dt=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}x^{2n+1}=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots$

Evaluate at $x=1$: $\dfrac{\pi}{4}=\tan^{-1}(1)=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{2n+1}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\ldots$

Since this is an alternating series, the truncation error satisfies $E_j < a_{j+1}$. So to determine a value of n which would guarantee an error less than 10^{-4}, we would solve

$$E_n < a_{j+1}=\frac{1}{2n+1}<10^{-4} \Rightarrow 2n+1>10^4 \Rightarrow 2n>10^4-1 \Rightarrow n>\frac{10^4-1}{2} \Rightarrow n=5000$$

In other words, the series does not converge very quickly. Fortunately, other formulae have been derived over the centuries.