Calculus 141, section 9.9 Taylor Series Completed

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Introduced in 9.1, and refined a little in 9.8, we have $f(x)=\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ for $|x| < R$, the Taylor polynomial for *f* on an open interval *I* containing 0. Two limitations to its usefulness exist, however. First of all, the domain of a function may be larger than the interval *I* over which the power series is defined, e.g. $f(x)=\frac{1}{1+x^2}$, for which *R* = 1, but which is defined and differentiable for all real numbers. Also, the requirement that *f* have derivatives of all orders on *I* means that not all functions have a Taylor series representation, e.g. $f(x)=|x|$ which is continuous everywhere but not differentiable at *x* = 0.

We now finish our investigation of Taylor polynomials by determining more general requirements under which the Taylor series for a function converges to that function for any *x*. For example, while we have derived Taylor series for sin *x* and cos *x*, we have not shown yet that they are valid for all values of *x*.

Example A: A classically used example, also discussed in the text, is the function $f(x)=\begin{cases} 0, & x=0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$.

While the Taylor series representation about *x* = 0 of this function converges for all values of *x*, it converges to *f*(*x*) only for *x* = 0.

Theorem 9.30, Taylor's Formula expanded about *x* = 0 (also called the Maclaurin Series), states: For *n* > 0 and if $f^{(n+1)}(x)$ exists for each *x* in an open interval *I* containing 0, then for each $x \neq 0$ in *I*, there is a number $0 < t_x < x$ such that $f(x)=\sum_{m=0}^{n}\left(\frac{f^{(m)}(0)}{m!}x^m\right)+\frac{f^{(n+1)}(t_x)}{(n+1)!}x^{n+1}$. The remainder $r_n(x)=\frac{f^{(n+1)}(t_x)}{(n+1)!}x^{n+1}$ is the Lagrange Remainder Formula. With an application of the Mean Value Theorem [since for *n* = 0, $f(x)-f(0)=f'(t_x)*(x-0)$], it is simple to show that for any *x* $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ if and only if $\lim_{n\to\infty} r_n = 0$.

Example B. Derive the Taylor expansion for *f*(*x*) = sin *x* and show that it converges to sin *x*.

Answer: $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Example C: Show that the Taylor polynomial generated by $f(x) = e^x$ at $x = 0$ converges to e^x for all *x*.

Example C extended: Calculate *e* with an error less than 10^{-6}. *Answer*: 2.718282

Sometimes we might want to find a Taylor series centered at a value for *x* other than 0, i.e. $\sum_{n=0}^{\infty} c_n (x-a)^n$. For this case, we have Theorem 9.31, Taylor's Theorem: For *n* > 0 and if $f^{(n+1)}(x)$ exists for each *x* in an open interval *I* containing *a*, for each $x \neq a$ there is a number $a < t_x < x$ such that

$f(x) = \sum_{m=0}^{n} \frac{f^{(m)}(a)}{m!}(x-a)^m + \frac{f^{(n+1)}(t_x)}{(n+1)!}(x-a)^{n+1}$. The remainder $r_n(x) = \frac{f^{(n+1)}(t_x)}{(n+1)!}(x-a)^{n+1}$ is the Lagrange Remainder Formula.

Example D: Find the Taylor series of ln *x* about *x* = 1, and determine its radius of convergence.

Answers: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$; (0, 2]