Strategies for Testing Series

Now that we have learned several tests for determining convergence/divergence of series, we will need to learn how to recognize which test to apply and how to apply the tests efficiently. The main strategy is to try to classify the series according to its form. As with integration, facility with evaluating series comes with practice and experience. For us beginners, the following list of strategies will help us learn to recognize which test to apply.

Suggested Strategies

1. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then use the **Test for Divergence** to show $\sum a_n$ diverges.

2. If the series is of the form $\sum \frac{1}{n^p}$, then it is a **p-series**, and it converges when $p > 1$ and diverges when $p \leq 1$.

3. If the series has the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$, then it is a **Geometric Series**. It converges if $|r| < 1$, and its sum is $\frac{\text{1st term}}{1-r}$. If $|r| \geq 1$, then the series diverges. Sometimes some algebraic manipulation is required to write the series into this form.

4. If the series has a form that is similar to a p-series $\left(\text{for example, } \sum_{n=1}^{\infty} \frac{1}{n^4+2}\right)$ or a geometric series $\left(\text{for example, } \sum_{n=1}^{\infty} \frac{1}{2+3^n}\right)$, then the **Comparison Test** should be considered. Recall that the Comparison Test may only be used on series with **positive** terms and says:

   (a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

   (b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Notice that the terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series.

5. If you feel that the Comparison Test should be applicable, but you can't get the inequalities to go in the right direction, then try the **Limit Comparison Test**, which says given two series $\sum a_n$ and $\sum b_n$ with **positive** terms,

$$\text{If } \lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } 0 < c < \infty\text{, then either both series converge or both series diverge.}$$

For example, the series $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ seems like it should behave the same as the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. However, the inequality $\frac{1}{n^2-1} > \frac{1}{n^2}$ is in the "wrong" direction, so we cannot use the Comparison Test.

5. If the series is of the form $\sum(-1)^n b_n$, then the **Alternating Series Test** should be considered. The Alternating Series Test says if

(i) The terms of the series are decreasing (i.e. $b_{n+1} \leq b_n$) for all n

and

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

6. Series that involve factorials or constants raised to the nth power are often evaluated using the **Ratio Test**, which says:

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the Ratio Test is inconclusive.

Keep in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all rational or algebraic functions of n, so the Ratio Test should not be used for these types of series.

7. If $a_n = f(n)$, where $\int_1^\infty f(x)dx$ is easily evaluated, then the **Integral Test** is a good choice. The Integral Test says if f is continuous, positive, and decreasing on $[1, \infty)$, then:

(a) If $\int_1^\infty f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

As with the Comparison Test, the terms of the series being tested must be **positive** if you wish to use the Integral Test.