Convergence of series – Tests and strategies

- **Integral test** -- For a series $\sum a_n$ with a_n *positive and decreasing*, construct the function $f(x)$ where $f(n) = a_n$.

  If $\int_1^\infty f(x)dx < \infty$ then the series converges. If $\int_1^\infty f(x)dx = \infty$ or does not exist, then the series diverges.
- **Comparison test** -- This test applies to series with *positive* terms.
  a) If there is a convergent series $\sum b_n$ and $a_n \le b_n$ for all n, then $\sum a_n$ converges.
  b) If there is a divergent series $\sum b_n$ and $a_n \ge b_n$ for all n, then $\sum a_n$ diverges.
- **Limit comparison test** -- Consider the series $\sum a_n$ and $\sum b_n$, both with *positive* terms. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, is finite and non-zero, then either both series converge, or they both diverge.
- **Alternating series test** -- This test applies to series of the form $\sum (-1)^n b_n$. If $b_{n+1} \le b_n$ for all n, ***and*** $\lim_{n\to\infty} b_n = 0$, then the series converges. Note that if the second part of the condition fails ($\lim_{n\to\infty} b_n \ne 0$) then the series diverges.
- **Absolute convergence** -- If $\sum |a_n|$ converges, then the series $\sum a_n$ also converges and we say that it converges absolutely. The reverse is not true: convergence *does not* imply absolute convergence.
- **Ratio test** -- For a series $\sum a_n$:
  a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series converges absolutely (so it converges).
  b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or the limit does not exist, then the series diverges.
  c) If the limit is equal to 1 the test is inconclusive.
- **Root test** -- For a series $\sum a_n$:
  a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series converges absolutely (so it converges).
  b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or the limit does not exist, then the series diverges.
  c) If the limit is equal to 1 the test is inconclusive.
- **Miscellaneous**
  a) If $\lim_{n\to\infty} a_n \ne 0$ for any series $\sum a_n$, then the series diverges.
  b) $\sum x^n$ is the geometric series and converges for any x satisfying $|x| < 1$. Then $\sum_{n=1}^\infty x^{n-1} = \sum_{n=0}^\infty x^n = \frac{1}{1-x}$.
  c) $\sum \frac{1}{n^p}$ converges for all $p > 1$ and diverges for all $p \le 1$.

Some strategies to keep in mind:

a) Usually the best test for an alternating series is the alternating series test. The ratio test might also be useful.
b) The ratio test is very handy when you have terms involving n[th] powers or factorials $(x^n, x^{2n-1}, 2^n, (2n)!, etc.)$.
c) When your terms involve polynomials in n (for example $a_n = \frac{n-1}{n^2+n}$), one of the comparison tests might do the trick. Look at the leading terms and compare with an appropriate series.
d) Be careful, if one of the tests fails, it doesn't mean that the series diverges unless the test specifically says so. Try a different test. This applies when the limit from the ratio or the root test is equal to 1.
e) It's useful to remember some examples for each of the tests. Also remember, $\sum \frac{1}{n} = \infty$ but $\sum \frac{(-1)^n}{n} < \infty$.