Stat 400

Solutions to Midterm 1 (spring 2010)

1(a)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(b)
$$E(X) = (1)(1/6) + (2)(1/6) + (3)(1/6) + (4)(1/6) + (5)(1/6) + (6)(1/6)$$
$$= \left(\frac{1}{6}\right)(1+2+3+4+5+6)$$
$$= \left(\frac{1}{6}\right)(21) = 21/6 = 7/2$$

(c)
$$E(X^2) = (1^2)\left(\frac{1}{6}\right) + (2^2)\left(\frac{1}{6}\right) + (3^2)\left(\frac{1}{6}\right) + (4^2)\left(\frac{1}{6}\right) + (5^2)\left(\frac{1}{6}\right) + (6^2)\left(\frac{1}{6}\right)$$
$$= \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)$$
$$= \frac{91}{6}$$

1(d)

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{91}{6} - \frac{49}{4} = \frac{182}{12} - \frac{147}{12}$$

$$= \frac{35}{12}$$

(e)

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{6}, & 1 \le x < 2 \\ \frac{2}{6}, & 2 \le x < 3 \\ \frac{3}{6}, & 3 \le x < 4 \\ \frac{4}{6}, & 4 \le x < 5 \\ \frac{5}{6}, & 5 \le x < 6 \\ 1, & x \ge 6 \end{cases}$$

(f) Just shift all the values of X with their probabilities one unit to the left.

y	0	1	2	3	4	5
$P(Y=y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

2.

(a) $$E(X) = \int_0^1 x\, 2(1-x)\,dx$$

$$= 2\left(\int_0^1 x\,dx - \int_0^1 x^2\,dx\right)$$

$$= 2\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = 2\left(\tfrac{1}{6}\right)$$

$$= \tfrac{1}{3}$$

So $V(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2$

(b) $$E(X^2) = \int_0^1 x^2\, 2(1-x)\,dx$$

$$= 2\left(\int_0^1 x^2\,dx - \int_0^1 x^3\,dx\right)$$

$$= 2\left(\tfrac{1}{3} - \tfrac{1}{4}\right) = 2\left(\tfrac{1}{12}\right) = \tfrac{1}{6}$$

(c) If $x \le 0$ then $F(x)=0$

If $0<x<1$ then

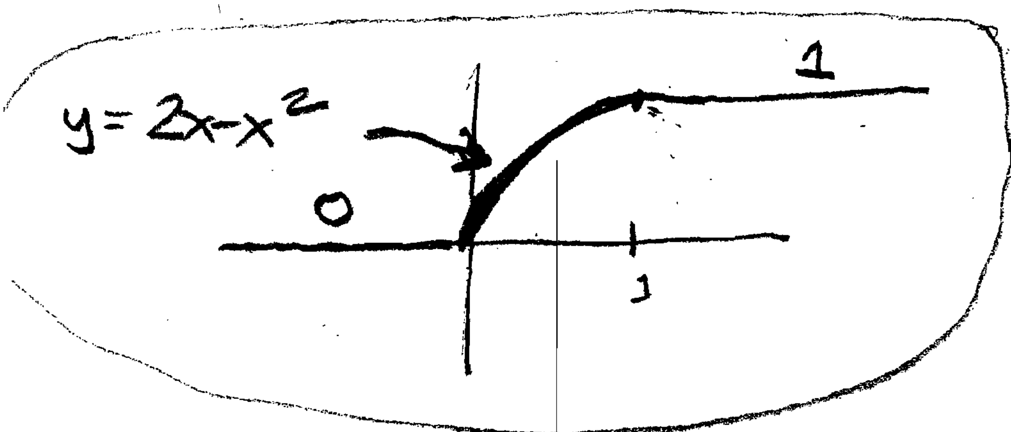
$$F(x)=\int_0^x 2(1-t)\,dt = 2\left(\int_0^x dt-\int_0^x t\,dt\right)$$

$$=2\left[(t)\Big|_{t=0}^{t=x}-\left(\frac{t^2}{2}\right)\Big|_{t=0}^{t=x}\right]$$

$$=2\left[x-\frac{x^2}{2}\right]=2x-x^2$$

So

$$F(x)=\begin{cases}0, & x<0\\ 2x-x^2, & 0\le x\le 1\\ 1, & x>1\end{cases}$$



← not required

2(d)' To find the median $\tilde{\mu}$ we solve

$$F(x) = \frac{1}{2} \quad (*)$$

The solution of (*) will be the median.

N.B. Now that any x such that $x < 0$
satisfies $F(x) = 0$ so it can't be the
solution of (*). Similarly, any
x such that $x > 1$ satisfies $F(x) = 1$
so it can't be the solution of (*)

Hence if x satisfies (*) then $0 < x < 1$
and so $F(x) = 2x - x^2$.

Hence (*) becomes

$$2x - x^2 = \frac{1}{2} \quad (**)$$

so $x^2 - 2x + \frac{1}{2} = 0$

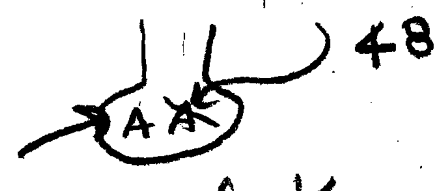
and $x = \frac{2 \pm \sqrt{4-2}}{2} = \frac{2 \pm \sqrt{2}}{2} = 1 \pm \frac{\sqrt{2}}{2}$

But $1 + \frac{\sqrt{2}}{2} > 1$ and $\tilde{\mu}$ must be
between 0 and 1 so

$$\tilde{\mu} = 1 - \frac{\sqrt{2}}{2}$$

~~2~~ 3. This is a special case of the hypergeometric distribution.

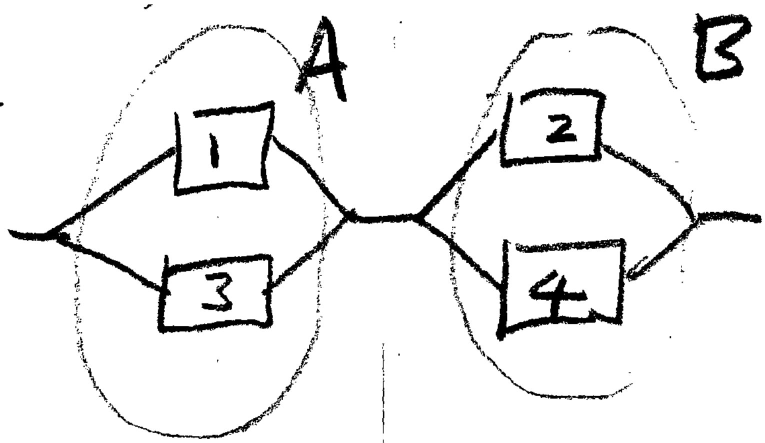
Imagine a container with
50 cards (because two have been dealt)
2 aces (because 2 aces " " ")
48 non-aces.



We pick 3 cards out of the container – so $\binom{50}{3}$ ways.

We want the number of ways to pick 2 aces so $\binom{2}{2}$ ways
AND
for each of those ways to pick one non-ace so $\binom{48}{1} = 48$ ways. So

$$P = \frac{\overset{1}{\binom{2}{2}}\binom{48}{1}}{\binom{50}{3}} = \frac{48}{\binom{50}{3}}$$

4. 

Lump 1 and 3 into A
and 2 and 4 into B

Define events

S = system works

A = A works (bad notation but it's OK)

B = B works

so (here is the point)

$$S = A \cap B$$

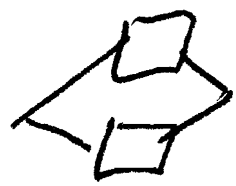
(system works $\Longleftrightarrow$ BOTH A and B work)

Since A and B don't have a component in common and the components function independently

not required

A and B are independent and

$$P(S) = P(A)P(B)$$

Since A and B are identical systems (both are )

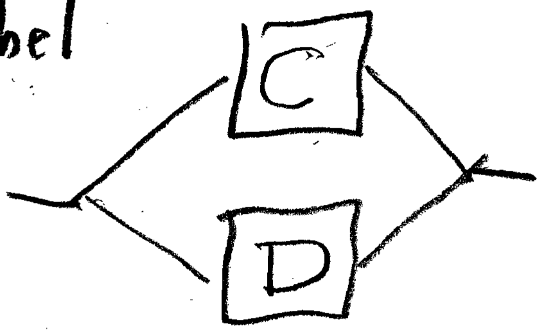
we have $P(A) = P(B)$

so

$$P(S) = P(A)^2$$

It remains to compute $P(A)$

Relabel



Define events

$C =$ C works

$D =$ D works

$A = C \cup D$ (A works $\Leftrightarrow$ EITHER C or D works)

so

$P(A) = P(C) + P(D) - P(C \cap D)$

Now C and D are independent so

$P(C \cap D) = P(C)P(D) = p^2$

so $P(A) = 2p - p^2$ and

$P(S) = (2p - p^2)^2$