1. (10 points) Suppose that, from past experience, the Math Dept. knows that, on average, 80% of those who enroll in a Calc I class pass (grade of C or better). The current enrollment in Calc I is 400 students.

1.a. Verify that this situation satisfies the 'rule of thumb" for calculating probabilities using a normal approximation.

The "rule of thumb" is that both np and nq must be greater than or equal to 10. In this case, $np = (400)(0.8) = 320$ and $nq = (400)(0.2) = 80$.

1.b. Use a normal approximation to a binomial distribution to determine the probability that more than 340 of the current students pass Calc I. (Don't forget the correction for continuity.)

$P[X > 340] = 1 - P[X \le 340]$; for normal approximation use continuity correction $1 - P[Y < 340.5]$

$$Z = \frac{Y-\mu}{\sigma} = \frac{Y-np}{\sqrt{npq}} \quad \Rightarrow \quad \frac{340.5 - 400(0.8)}{\sqrt{400(0.8)(0.2)}} \approx 2.56$$

$$1 - P(-\infty < Y < 340.5) = 1 - P(-\infty < Z < 2.56) = 1 - \Phi(2.56) = 1 - 0.9948 = 0.0052$$

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2. (12 points) Reliability studies use an exponential probability distribution to express the expected life time of a piece of machinery. A production machine in a factory has an expected lifetime of 5 years

2.a. State the probability density function $f(t)$ for random variable T = number of years until the machine breaks down.

Since $E(X) = 5$, we identify $\lambda = \frac{1}{5} = 0.2$.

$$f(t) = \begin{cases} 0 & -\infty < t < 0 \\ 0.2e^{-0.2t} & 0 \le t < \infty \end{cases}$$

2.b. Determine the cumulative distribution function for this piece of machinery.

$$\int_0^t 0.2e^{-0.2x}\, dx = \left[-\frac{0.2}{0.2}e^{-0.2x}\right]_0^t = -e^{-0.2t} + 1$$

$$F(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1 - e^{-0.2t} & 0 \le t < \infty \end{cases}$$

2.c. Find the probability that a new machine will last at least 6 years before it breaks down. (Think carefully about what this question is asking, and how it relates to part b.) Give both an exact value answer and an approximation rounded to 4 decimal places.

$$P(T \ge 6) = 1 - F(6) = 1 - \left(1 - e^{-0.2(6)}\right) = e^{-1.2} \approx 0.3012$$

2.d. Find the probability that a machine which is 3 years old will last at least another 6 years before it breaks down. (Think carefully about what this question is asking, and how it relates to part c. You shouldn't have to spend a long time on this question.) Give both an exact value answer and an approximation rounded to 4 decimal places.

Since an exponential probability distribution has the property that it is "memoryless", the fact that the machine hasn't broken down yet doesn't change the probabilities involved in how much longer it will take before it does break down, so $P(T \ge 6) = 1 - F(6) = 1 - \left(1 - e^{-0.2(6)}\right) = e^{-1.2} \approx 0.3012$.

It is incorrect to calculate $[1 - F(9)] - [1 - F(3)] = F(3) - F(9)$. The correct calculation would involve a conditional probability $P(T \ge 9 \mid T \ge 3) = \frac{P(T\ge 9 \cap T \ge 3)}{P(T \ge 3)} = \frac{P(T \ge 9)}{P(T \ge 3)} = \frac{1-F(9)}{1-F(3)} = \frac{e^{-1.8}}{e^{-0.6}} = e^{-1.2}$.

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3. (18 points) Two continuous random variables X and Y have joint probability density function

$$f(x, y) = \begin{cases} kxy & 0 \le x \le 1, 1 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

3.a. Determine the value of k.

$$\int_1^2 \int_0^1 kxy\, dx\, dy = \int_1^2 ky\left[\frac{x^2}{2}\right]_0^1 dy = \int_1^2 \frac{ky}{2}\, dy = \left[\frac{ky^2}{4}\right]_1^2 = \frac{k(2^2)}{4} - \frac{k(1^2)}{4} = \frac{3k}{4} = 1 \quad \Rightarrow \quad k = \frac{4}{3}$$

3.b. Find Cov (X,Y). (You may find the work for this one easier if you use k instead of a number value, the put in your answer from part a. when it comes time to do the calculations.)

$$E(XY) = \int_1^2 \int_0^1 xy * kxy\, dx\, dy = \int_1^2 \int_0^1 kx^2y^2\, dx\, dy = \int_1^2 \left[\frac{1}{3}kx^3y^2\right]_0^1 dy = \int_1^2 \frac{1}{3}ky^2\, dy$$

$$= \left[\frac{1}{9}ky^3\right]_1^2 = \frac{8k}{9} - \frac{k}{9} = \frac{7k}{9} = \frac{7}{9} * \frac{4}{3} = \frac{28}{27}$$

$$f_X(x) = \int_1^2 kxy\, dy = \left[\frac{kxy^2}{2}\right]_1^2 = \frac{4kx}{2} - \frac{kx}{2} = \frac{3kx}{2} = \frac{4}{3} * \frac{3x}{2} = 2x$$

$$\mu_X = \int_0^1 x(2x)\, dx = \int_0^1 2x^2\, dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}$$

$$f_Y(y) = \int_0^1 kxy\, dx = \left[\frac{kx^2y}{2}\right]_0^1 = \frac{ky}{2} - 0 = \frac{ky}{2} = \frac{4}{3} * \frac{y}{2} = \frac{2y}{3}$$

$$\mu_Y = \int_1^2 y\left(\frac{2}{3}y\right) dx = \int_0^1 \frac{2}{3}y^2\, dx = \left[\frac{2}{9}y^3\right]_1^2 = \frac{16}{9} - \frac{2}{9} = \frac{14}{9}$$

$$\text{Cov}(X, Y) = E(XY) - \mu_X * \mu_Y = \frac{28}{27} - \frac{2}{3} * \frac{14}{9} = 0$$

3.c. Are random variables X and Y independent?

It is insufficient to state "Cov (X,Y) = 0 implies independence" because this conditional logic statement is not true. You must show, "Since $f_X(x) * f_Y(y) = 2x * \frac{2y}{3} = \frac{4xy}{3} = f(x, y)$, X and Y are independent."

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4. (10 points) An experiment has the probability distribution given in the table below. A sample of size 36 is randomly selected.

X	0	1	2	3
$P(X)$	0.4	0.2	0.3	0.1

4.a. What is the probability that a single randomly-chosen value of X is less than 1.5?

$$P(X < 1.5) = P(X = 0) + P(X = 1) = 0.4 + 0.2 = 0.6$$

4.b. What is the probability that the sample mean is less than 1.5?

$$E(\overline{X}) = E(X) = 0(0.4) + 1(0.2) + 2(0.3) + 3(0.1) = 0 + 0.2 + 0.6 + 0.3 = 1.1$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 0^2(0.4) + 1^2(0.2) + 2^2(0.3) + 3^2(0.1) - [1.1]^2 = 1.09$$

$$\sigma(X) = \sqrt{1.09}$$

The sample size n = 36 satisfies our rule-of-thumb criterion for invoking the Central Limit Theorem.

$$Z = \frac{\overline{X} - \mu_X}{\sigma_X / \sqrt{n}} \quad \Rightarrow \quad \frac{1.5 - 1.1}{\sqrt{\frac{1.09}{36}}} \approx 2.30, \; P(\overline{X} \le 1.5) = P(Z \le 2.30) = 0.9893$$
