1. (a)

$$P(X=0)=\frac{1}{4}+\frac{1}{4}+\frac{1}{8}=\frac{5}{8};\quad P(X=1)=\frac{1}{4}+0+\frac{1}{8}=\frac{3}{8}$$

$$P(Y=0)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2};\quad P(Y=1)=\frac{1}{4}+0=\frac{1}{4};\quad P(Y=2)=\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

1. (b) No. counterexample: $P[(x,\ y)=(0,\ 0)]=\frac{1}{4}\neq P_X(X=0)*P_Y(Y=0)=\frac{5}{8}*\frac{1}{2}=\frac{5}{16}$

Any correct counterexample will do.

1. (c)

$W = X+Y$	0 (0 + 0)	1 (0 + 1 or 1 + 0)	2 (0 + 2 or 1 + 1)	3(1 + 2)
$P(W=w)$	$\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$	$\frac{1}{8}+0=\frac{1}{8}$	$\frac{1}{8}$

1. (d) $E(XY)=(0)\left(\frac{1}{4}\right)+(0)\left(\frac{1}{4}\right)+(0)\left(\frac{1}{8}\right)+(0)\left(\frac{1}{4}\right)+(1)(0)+(2)\left(\frac{1}{8}\right)=\frac{1}{4}$

$$E(X)=(0)\left(\frac{5}{8}\right)+(1)\left(\frac{3}{8}\right)=\frac{3}{8};\quad E(Y)=(0)\left(\frac{1}{2}\right)+(1)\left(\frac{1}{4}\right)+(2)\left(\frac{1}{4}\right)=\frac{3}{4}$$

$$Cov(X,Y)=\frac{1}{4}-\frac{3}{8}*\frac{3}{4}=\frac{8}{32}-\frac{9}{32}=-\frac{1}{32}=-0.03125$$

1. (e) $E\left(X^2\right)=\left(0^2\right)\left(\frac{5}{8}\right)+\left(1^2\right)\left(\frac{3}{8}\right)=\frac{3}{8};\quad \sigma_X=\sqrt{\frac{3}{8}-\left(\frac{3}{8}\right)^2}=\sqrt{\frac{24}{64}-\frac{9}{64}}=\frac{\sqrt{15}}{8}(\approx 0.48412291827592\ldots)$

$$E\left(Y^2\right)=\left(0^2\right)\left(\frac{1}{2}\right)+\left(1^2\right)\left(\frac{1}{4}\right)+\left(2^2\right)\left(\frac{1}{4}\right)=\frac{5}{4};\quad \sigma_y=\sqrt{\frac{5}{4}-\left(\frac{3}{4}\right)^2}=\sqrt{\frac{20}{16}-\frac{9}{16}}=\frac{\sqrt{11}}{4}(\approx 0.82915619758886\ldots)$$

$$Corr(X,Y)=\rho_{X,Y}=-\frac{1}{32}\div\left(\frac{\sqrt{15}}{8}*\frac{\sqrt{11}}{4}\right)=-\frac{1}{\sqrt{165}}(\approx -0.07784989441616\ldots)$$

(20 points)

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2. (a) An important key here is that part (a) asks for ***<u>binomial</u>*** coefficients and fractions.

P(more than 50) = P(51 or 52 or 53 or … or 99 or 100)

$$P(X>50)=P(X\geq 51)=\sum_{x=51}^{100}\binom{100}{x}\left(0.5^x\right)\left(0.5^{100-x}\right)\quad or\quad \sum_{x=51}^{100}\binom{100}{x}\left(0.5^{100}\right)$$

alternate: $P(X>50)=1-P(X\leq 50)=1-\sum_{x=0}^{50}\binom{100}{x}\left(0.5^x\right)\left(0.5^{100-x}\right)\quad or\quad 1-\sum_{x=0}^{50}\binom{100}{x}\left(0.5^{100}\right)$

2. (b) three methods:

easiest: $E(X) = np = 100(0.5) = 50$; normal distribution is symmetric, so $P(X > 50) = P(X > E(X)) = 0.5$

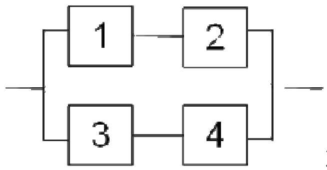
second easiest (no continuity correction): $z=\frac{51-100(0.5)}{\sqrt{100(0.5)(0.5)}}=\frac{1}{\sqrt{25}}=0.2$

$$P(X\geq 51)=1-P(X<51)=1-P(z<0.2)=1-0.5793=0.4207$$

most complicated (continuity correction): $z=\frac{50.5-100(0.5)}{\sqrt{100(0.5)(0.5)}}=\frac{0.5}{\sqrt{25}}=0.1$

$$P(X\geq 51)\ \Rightarrow\ 1-P(X<50.5)=1-P(z<0.1)=1-0.5398=0.4602$$

(10 points)



3. components function independently; exponentially distributed with parameter λ

3. (a) Define ($X_i = t$) = component i fails at time t. You were given $P(X > t) = e^{-\lambda t}$ (at the bottom of the first page)

Define ($A = t$) = either 1 or 2 fails at time t, so

$P(A > t) = P([X_1 > t] \cap [X_2 > t]) = P(X_1 > t) * P(X_2 > t) = e^{-\lambda t} * e^{-\lambda t} = e^{-2\lambda t}$ and $P(A \le t) = 1 - e^{-2\lambda t}$.

We would get the same result for B, since the structure of B is identical to that of A.

Define ($S = t$) = system fails at time t . So $(S \le t) = (A \le t) \cap (B \le t)$. In words: the system fails if both A and B fail. [This is one of deMorgan's Laws: $S' = (A \cup B)' = A' \cap B'$.]

Since components are independent we have $P(S \le t) = P(A \le t) * P(B \le t) = \left(1 - e^{-2\lambda t}\right)^2$.

$P(S \le t)$ is, by definition, the cumulative distribution function $F(t)$, from time 0 to time t, so formally

$$F(t) = \begin{cases} 0 & -\infty < t < 0 \\ \left(1 - e^{-2\lambda t}\right)^2 & 0 \le t < \infty \end{cases}$$

3. (b) The cdf is the integral of the pdf, and the pdf is the derivative of the cdf. Chain rule is needed.

$$\frac{d}{dt}\left(1 - e^{-2\lambda t}\right)^2 = 2\left(1 - e^{-2\lambda t}\right)^1 * \frac{d}{dt}\left(1 - e^{-2\lambda t}\right) = 4\lambda t\left(1 - e^{-2\lambda t}\right)$$

The probability density function is $f(t) = \begin{cases} 4\lambda t\left(1 - e^{-2\lambda t}\right) & 0 \le t < \infty \\ 0 & \text{otherwise} \end{cases}$

(10 points)

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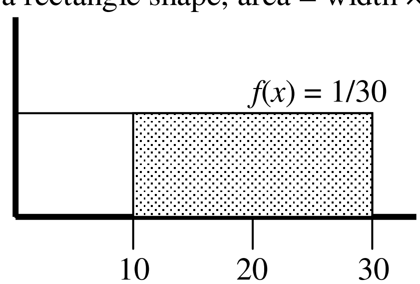
4. (a) easy way: A uniform distribution density function graph is a horizontal line; the "area under the curve" is a rectangle shape, area = width × height; entire area must = 1, so since "width" = 30 minutes, "height" = 1/30.



$$P(X > 10) = (30 - 10) * \frac{1}{30} = \frac{2}{3}$$

slightly harder way: $\int_{10}^{30} \frac{1}{30}\,dt = \left[\frac{1}{30}t\right]_{10}^{30} = 1 - \frac{1}{2} = \frac{2}{3}$

4. (b) This is conditional probability.

$$P[(X > 20) \mid (X > 10)] = \frac{P[(X > 20) \cap (X > 10)]}{P[X > 10]}$$

But note that $P[(X > 20) \cap (X > 10)] = P[X > 20]$!

So, $P[(X > 20) \mid (X > 10)] = \dfrac{P[(X > 20) \cap (X > 10)]}{P[X > 10]} = \dfrac{P[X > 20]}{P[X > 10]} = \dfrac{1/3}{2/3} = \dfrac{1}{2}$ (Visually from the picture above: If you've already waited 10 minutes, "another 10 minutes" is ½ of the remaining "area under the curve".)

Important: Uniform probability is not "memoryless", so the fact that you have waited 10 minutes already is relevant. P("another 10 minutes from now") ≠ P("10 minutes from time zero").

(10 points)
