Final Exam Spring 2011

Solutions

1. The number of possible birthdays $= (365)^n$

The number of distinct birthdays for n people

$$= (365)(364)\cdots(365-n+1)$$

$$= \frac{(365)!}{(365-n)!}$$

So $$P(\text{all birthdays distinct}) = \frac{\left(\frac{(365!)}{(365-n)!}\right)}{(365)^n}$$

$$= \frac{(365)!}{(365-n)!\,(365)^n}$$

So $B_n = P(\text{at least two students have the same birthday})$

$$= 1 - \frac{(365)!}{(365-n)!\,(365)^n}$$

2

(a) $E(X) = \int_0^1 x \, 6x^5 dx = 6\int_0^1 x^6 dx = \frac{6}{7}$

(b) $E(X^2) = \int_0^1 x^2 \, 6x^5 dx = 6\int_0^1 x^7 dx = \frac{6}{8}$

(c) $V(X) = \left(\frac{6}{8}\right) - \left(\frac{6}{7}\right)^2$

(d) $F(x) = \begin{cases} 0, & x < 0 \\ x^6, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$

(e) To find the median $\tilde{\mu}$ we solve

$$F(\tilde{\mu}) = \frac{1}{2}$$

that is $$(\tilde{\mu})^6 = \frac{1}{2}$$

so $$\tilde{\mu} = \left(\frac{1}{2}\right)^{\frac{1}{6}}$$

3. Let X be the number of heads so $X \sim Bin(100, \frac{1}{2})$ and we want $P(X > 60)$.

Now $E(X) = np = (100)(\frac{1}{2}) = 50$

and $V(X) = npq = (100)(\frac{1}{2})(\frac{1}{2}) = 25$

So we let Y be a normal random variable with mean 50 and variance 25 (so $Y \sim N(50, 25)$) and apply the Normal Approximation to the binomial to deduce

$$P(X > 60) \approx P(Y > 60)$$

$$= P\left(\frac{Y-50}{5} > \frac{60-50}{5}\right)$$

$$= P(Z > 2)$$

$$= 1 - P(Z \le 2)$$

$$= 1 - .9772 = .0228$$

However we can arrive at a different answer as follows

$$P(X \leq 60) = 1 - P(X < 60) = 1 - P(X \leq 59)$$

$$\approx 1 - P(Y \leq 59) = 1 - P\left(\cancel{\frac{Y-50}{5}} \leq \frac{59-50}{5}\right)$$

$$= 1 - P(Z \leq \tfrac{9}{5}) = 1 - P(Z \leq 1.8)$$

$$= 1 - .9641 = .0359$$

from text, it wasn't on my table.

Finally, if we did use the correction for continuity we would get

$$P(X \geq 60) \approx P(Y \leq 59.5) = 1 - P(Y \leq 59.5)$$

$$= 1 - P(Z \leq 1.9) = .0287.$$

using the second method above

$$P(X \geq 60) = 1 - P(X \leq 59) \approx 1 - P(X \leq 59.5) = .0287$$

THE SAME AS ON THE LINE ABOVE

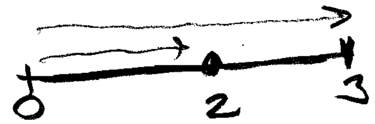
If you go to the web (binomial probability calculator) you get

$$P(X \geq 60) = .0284$$

[illegible] OF THE ABOVE WILL BE COUNTED AS CORRECT

4. Let $(X=t) = ($the Ford fusion breaks down after t years$)$

(a)



We put $A = (X \geq 3)$ and $B = (X \geq 2)$

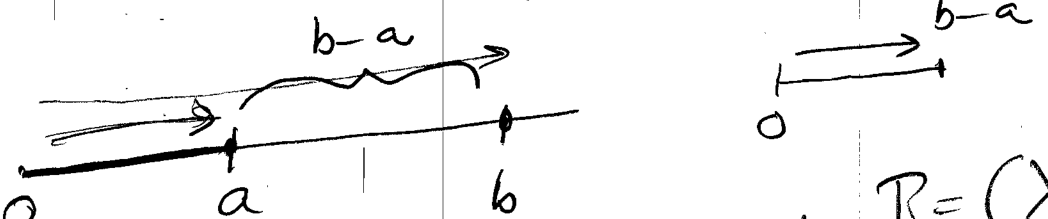
so we want

$$P(X \geq 3 \mid X \geq 2) = P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

But $A \subseteq B$ so $A \cap B = A$ so

$$P(X \geq 3 \mid X \geq 2) = \frac{P(X \geq 3)}{P(X \geq 2)} = \frac{e^{-\frac{3}{5}}}{e^{-\frac{2}{5}}} = e^{-\frac{1}{5}}$$

(b)



Now we put $A = (X \geq b)$ and $B = (X \geq a)$

As before we have $A \subseteq B$ so $A \cap B = A$ and

$$P(X \geq b \mid X \geq a) = \frac{P(X \geq b)}{P(X \geq a)} = \frac{e^{-b/5}}{e^{-a/5}} = e^{-\frac{(b-a)}{5}}$$

But $P(X \geq b-a) = e^{-\frac{(b-a)}{5}}$

(c) Under the assumption that X is exponential I would rather buy the cheaper car.

The exponential model is a bad one — everybody knows cars wear out, a new car will last longer than a used one.

5.

X \ Y	0	1	2	3	
0	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{1}{32}$	$\frac{1}{4}$
1	$\frac{2}{32}$	$\frac{6}{32}$	$\frac{6}{32}$	$\frac{2}{32}$	$\frac{1}{2}$
2	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{1}{32}$	$\frac{1}{4}$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

(a)

X	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Y	0	1	2	3
$P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(b) Yes they are independent.

(c) W takes the values 0, 1, 2, 3, 4, 6

W	0	1	2	3	4	6
$P(W=w)$	$\frac{11}{32}$	$\frac{6}{32}$	$\frac{9}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{1}{32}$

(d) W takes the value 0, 1, 2

W	0	1	2
$P(W=w)$	$\frac{11}{32}$	$\frac{17}{32}$	$\frac{4}{32}$

(e) $Cov(X,Y)=0$ because X and Y are independent

5. $\boxed{X \sim N(\mu, 1)} \longrightarrow 2, 1, 3, 1, 3, 2, 2, 3, 1$

(a) $\overline{x} = \frac{18}{9} = 2$

(b) The formula for the $100(1-\alpha)\%$ confidence interval is

$$\left(\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

We have $(100)(1-\alpha)\% = 90\%$ so $1-\alpha = .9$ so $\alpha = .1$ so $\alpha/2 = .05$

so from the table $z_{\alpha/2} = z_{.05} = 1.645$

Hence the radius of the confidence interval is

$$z_{\alpha/2}\frac{\sigma}{\sqrt{9}} = (1.645)\frac{1}{3} = \frac{(1.645)}{3} = .55$$

THIS OKAY FOR THE ANSWER

so the interval is

$$\left(2 - \frac{1.645}{3}, 2 + \frac{1.645}{3}\right)$$

$$= (2 - .55, 2 + .55)$$

$$= (1.45, 2.55)$$

8

7. We have to prove

$$P\left(\mu \in \left(-\infty, \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)\right) = 1-\alpha$$

$$LHS = P\left(\cancel{-\infty <} \mu, \mu < \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)$$ ← you can leave this step out

$$= P\left(\mu < \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}\right)$$

swap

$$= P\left(-z_\alpha \frac{\sigma}{\sqrt{n}} < \overline{X} - \mu\right)$$ unnecessary

divide

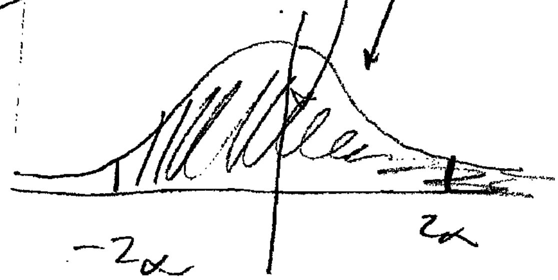
$$= P\left(-z_\alpha < \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

make the sign of the Z

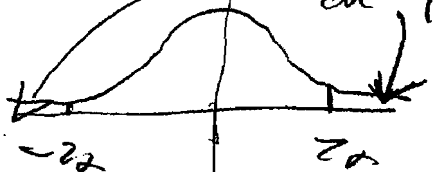
$$= P(-z_\alpha < Z)$$

≐ the shaded area

one point for this picture



these two areas are equal



one point for this picture

But the unshaded area is α by symmetry and the definition of z_α.

The total area is one so

The shaded area $= 1-\alpha$

$\square$