**Stat 400** **FINAL EXAM** **May 12, 2011**

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1. Suppose our class has $n$ people in it. Write down an expression in terms of $n$ for the probability that at least two students will have the same birthday. Assume that a year has 365 days; that is, ignore leap years.

(10 points)

2. Let $X$ be a continuous random variable with the probability density function

$$f(x) = 6x^5, 0 \leq x \leq 1$$

and

$$f(x) = 0, \ otherwise.$$

(a) Find E(X).

(b) Find $E(X^2)$.

(c) Find V(X).

(d) Find $F(x)$, the cumulative distribution function of $X$.

(e) Find the median of $X$.

(20 points)

3. Use the normal approximation to the binomial to give a number that approximates the true probability of obtaining 60 or more heads in 100 tosses of a fair coin (don't use the correction for continuity).
The relevant entries from the tables of normal probabilities taken from the front flap of the text are contained in the part of the table reproduced immediately below.

(10 points).

**Table A.3 Standard Normal Curve Areas**

| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .85331 | .8554 | .8577 | .8599 | .8621 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |

4. Suppose the lifetime $X$ in years of a Ford Fusion is exponentially distributed with parameter 1/5. We recall that this means that the probability mass function of $X$ is given by

$$f(t) = \begin{cases} \frac{1}{5}e^{-\frac{1}{5}t}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

For the purposes of this problem you will need only $P(X \geq t) = e^{-\frac{1}{5}t}$.
In (a) and (b) you will need to use the formula

$$P(A|B) = (P(A) \cap P(B))/P(B).$$

(a) Suppose you have owned a Ford Fusion two years or more . What is the probability it will last at least three years (so at least one more year)?

(b) Prove the general result that if $b > a$ and a Ford Fusion has lasted at least $a$ years then the probability that it will last at least $b$ years (so at least $b - a$ more years) is the same as the probability that a *brand new* Ford Fusion will last at least $b - a$ years (this is called the "memorylessness" of the exponential distribution).

(c) In the light of (b) would you rather buy a *brand new* Ford Fusion for \$ 20,000 or a five year old Ford Fusion for \$ 5,000 (ignore aesthetic considerations)? Do you think that the exponential distribution is a good model for the probable lifetimes of automobiles?

(10 points, part (c) is worth two points)

5. Suppose $X$ and $Y$ are random variables defined on the same sample space with the following joint probability mass function.

| X \ Y | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1/32 | 3/32 | 3/32 | 1/32 |
| 1 | 2/32 | 6/32 | 6/32 | 2/32 |
| 2 | 1/32 | 3/32 | 3/32 | 1/32 |

(a) Compute the probability mass functions of the random variables $X$ and $Y$.

(b) Are $X$ and $Y$ independent?
(c) Compute the probability mass function of the random variable $W = XY$.
(d) Compute the probability mass function of the random variable $W = min(X, Y)$.
The function $min(x, y)$ assigns the smaller number of the pair $(x, y)$ to that pair, so $min(1, 2) = 1$ and $min(1, 1) = 1$.
(e) What is $Cov(X, Y)$ (use your answer from (b) to avoid making a long computation).

(30 points)

6. Suppose a sample of size 9 from a normal distribution with variance 1 yielded 2,1,3,1,3,2,2,2,2.
(a) Give a point estimate for $\mu$ .
(b) Construct a 90% confidence interval for $\mu$.
You will need the following table

| $\alpha$ | .1 | .05 | .025 | .01 | .005 |
|---|---|---|---|---|---|
| $z_\alpha$ | 1.28 | 1.645 | 1.96 | 2.33 | 2.58 |

(10 points)

7. Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution of mean $\mu$ and variance $\sigma^2$. Suppose the variance $\sigma^2$ is known. Prove that the random interval $(-\infty,\ \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}})$ is a $100(1-\alpha)\%$ confidence interval for the mean $\mu$. You will need the following

**Theorem** The random variable $Z = (\overline{X} - \mu)/(\frac{\sigma}{\sqrt{n}})$ has standard normal distribution.

(10 points)