ON THE DYNAMICS OF LIQUID–VAPOR PHASE TRANSITION
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Abstract. We consider a multidimensional model for the dynamics of liquid–vapor phase transitions. In the present context liquid and vapor are treated as different species with different volume fractions and different molecular weights. The model presented here is a prototype of a “binary fluid mixture,” and is formulated by the Navier Stokes equations in Euler coordinates. This system takes now a new form due to the choice of rather complex constitutive relations that can accommodate appropriately the physical context. The setting of the problem presented in this work is motivated by physical considerations (cf. Fan [18], Fan and Slemrod [19], Slemrod [34]). The transport fluxes satisfy rather general constitutive laws, the viscosity and heat conductivity depend on the temperature, the pressure law is a nonlinear function of the temperature depending on the mass density fraction of the vapor (liquid) in the fluid as well as the molecular weights of the individual species. The existence of globally defined weak solutions of the Navier-Stokes equations for compressible fluids is established by using weak convergence methods, compactness and interpolation arguments in the spirit of [22] [32], (see also [12], [13]).

Key words. Navier-Stokes system, compressible fluids, liquid-vapor phase transitions, binary fluid mixtures, oscillation and concentration phenomena.

AMS subject classifications. 35L65, 76N10, 35B45.

1. Introduction. A multidimensional model is presented for the liquid–vapor phase transition formulated by the Navier Stokes equations in Euler coordinates. In the present context, these equations express the conservation of mass, the balance of momentum and entropy and the species conservation equation.

Consider a pure fluid which exhibits liquid-vapor phase changes. In the macroscopic description adopted here \( \rho = \rho(t, x) \) denotes the density of the mixture, \( u = u(t, x) \) its average velocity, \( \theta = \theta(t, x) \) the temperature, while \( f_1 = f_1(x, t) \) denotes the mass density fraction of vapor in the fluid at time \( t \in \mathbb{R} \) and at the spatial position \( x \in \Omega \subset \mathbb{R}^N, N = 3 \). These macroscopic variables provide a precise characterization of the state of the mixture, which in the present context consists of the species vapor and liquid. The species conservation equation is given by

\[
\partial_t (\rho f_1) + \text{div}(\rho f_1 u) + \text{div} F_1 = w_1.
\]

Here and in what follows,

\( \cdot \) \( f_i(x, t) \) is the volume fraction of the \( i \) species. Vapor and liquid fill up the space, namely

\[
f_1 + f_2 = 1.
\]  

(1.1)

\( \cdot \) \( \rho_i(x, t) \) is the density of the \( i \) species

\[
\rho_i = \rho f_i, \quad \rho = \rho_1 + \rho_2.
\]  

(1.2)

\( \cdot \) \( w_i \) is the reaction rate function denoting the mass of the \( i \)-species produced per unit volume per time unit. In accordance with the

\[
\text{conservation of mass:} \quad \partial_t \rho + \text{div}(\rho u) = 0,
\]

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we have
\[ w_1 + w_2 = 0. \] (1.3)

The function \( w_1 \) is the so-called reaction rate function which governs the growth of vapor in the fluid. In general, \( w_1 \) consists of two parts, \( w_{\text{growth}} \) and \( w_{\text{nucleation}} \). The former accounts for the creation of nuclei of new phase, while the latter describes the subsequent growth of these nuclei,
\[ w_1 = w_{\text{growth}} + w_{\text{nucleation}}. \] (1.4)

The \textit{momentum conservation equation} can be written
\[ \text{balance of momentum: } \partial_t (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \text{div} \mathbb{S}, \]
where \( \rho \) is the total mass density, \( \mathbf{u} \otimes \mathbf{u} \) is the tensor product of velocity vectors, \( p \) denotes the pressure, while \( \mathbb{S} \) is the viscous stress tensor.

Finally the \textit{energy conservation equation} reads
\[ \text{balance of energy: } \partial_t (\rho e) + \text{div} (\rho \mathbf{u} e) + \text{div} \mathbf{q} = \mathbb{S} : \nabla \mathbf{u} - p \text{div} \mathbf{u} \]

The physical properties of the mixture are expressed through \textit{constitutive relations}, which specify the relation of the viscous stress tensor \( \mathbb{S} \), the heat flux \( \mathbf{q} \), the pressure \( p \) and the internal energy \( e \) to the \textit{macroscopic variables}. The hypothesis on the constitutive laws and on the special features of the model are the following.

\textbf{H.1} The viscous stress tensor \( \mathbb{S} \) is given by the \textit{Newton’s viscosity formula}
\[ \mathbb{S} = \mu(\theta) \left( \nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \text{div} \mathbf{u} I \right) + \zeta(\theta) \text{div} \mathbf{u} I, \] (1.5)
where the \textit{shear viscosity} \( \mu \) and the \textit{bulk viscosity} \( \zeta \) are supposed to be nonnegative and continuously differentiable functions of the absolute temperature.

\textbf{H.2} The pressure in the portion occupied by the \( i \text{-th} \) component \( p_i = p(\rho_i, \theta) \) depends in a crucial way on the volume fraction \( f_i \). In the case where the fluid is in \textit{pure} phase, either (pure) vapor or (pure) liquid the pressure can be determined quite accurately through experiments. Let us denote by \( p_1 = p(\rho_1, \theta) \) and \( p_2 = p(\rho_2, \theta) \) the pressure of pure vapor phase and liquid phase respectively, with \( \rho_i \) the density of the \( i \)-component and \( \theta \) the temperature of the mixture (see Figure 1.1). The density at which the pressure of the individual species coincide, namely \( p_1(m, \theta) = p_2(M, \theta) = p_0 \) is known as \textit{Maxwell equilibrium density} and corresponds to \( \rho = m(\theta), \rho = M(\theta) \) in Figure 1.1. At this density, both species liquid and vapor coexist, are (as we say) \textit{in equilibrium} and in this case the pressure can be measured in experiments. On the other hand, the region that corresponds to rapid \textit{phase transition}, namely the area where the material instantly decomposes into liquid or vapor or their mixtures (\( \rho \in (a, b) \) in Figure 1.1) is highly unstable. As a consequence this part of the so-called \textit{van der Waals} pressure curve (the dashed line in Figure 1.1) cannot be measured in experiments and it is regarded as artificial (see also the discussion in Fan [18], Fan and Slemrod [19], Slemrod [34] and the references therein).
As in Fan [18], we proceed considering only the physically relevant part (the part observed in experiments) and we extend each component of the function $p(\rho, \theta)$ continuously as shown in Figure 1.2. This extension may appear at first a bit artificial, however this part of the pressure does not really affect the outcome as indicated by physical experiments. In other words, we treat liquid and vapor as different species and specify the transition of the phases by the reaction rate equation (1.4).
Taking the above discussion into consideration, we assume the *Dalton-type law* for the pressure of the mixture, namely

\[ p = p_R + p_B + p_e \]  

(1.6)

where the term \( p_B \) satisfies the *Boyle’s law* for the individual species

\[ p_B = \frac{\rho_1 R \theta}{m_1} + \frac{\rho_2 R \theta}{m_2}, \]  

(1.7)

with \( m_i \) denotes the molecular weight of the \( i \)th component, \( p_e = p_e(\rho) \) is the so-called *elastic pressure* whose properties will be discussed in the sequel, while \( p_R \) accounts for *radiation effects*. We remark that the *elastic pressure* may include higher order terms as in the *Beattie-Bridgman model*, where the state equation for the pressure includes an *elastic* component of the form

\[ p_e(\rho) = \beta_1 \rho^2 + \beta_2 \rho^3 + \beta_3 \rho^4, \]

for appropriate constants \( \beta_i \), [1], [22].

The point of view presented here takes into consideration both the classical as well as the *quantum* aspects associated with the fluid. In the quantum case, the presence of the photons effects the total pressure \( p \) in the fluid. As a result the pressure function now includes an additional (radiation) component \( p_R \), which is related to the absolute temperature \( \theta \) through the *Stefan-Boltzmann law*

\[ p_R = \frac{a}{3} \theta^4, \quad \text{with} \quad a > 0 \quad a \text{ constant}. \]

The underlying assumption here (cf. [14], [27], [33]) is that the *high temperature radiation*, is at thermal equilibrium with the fluid. Analogously, standard thermodynamic relations require that the specific internal energy of the fluid be also augmented, by the term

\[ e_R = e_R(\rho, \theta) = \frac{a}{\rho} \theta^4. \]

We remark that *radiation effects* are of particular interest in *astrophysical plasma models* [4].

The pressure therefore satisfies, in the present context, the general law

\[ p = \frac{a}{3} \theta^4 + \frac{\rho_1 R \theta}{m_1} + \frac{\rho_2 R \theta}{m_2} + p_e(\rho). \]

Taking into consideration (1.1), (1.2) the pressure law now takes the form

\[ p = \frac{a}{3} \theta^4 + \frac{\rho}{m_2} R \theta + R \left(1 - \frac{m_1}{m_2}\right) \frac{\rho_1 \theta}{m_1} + p_e(\rho). \]

The molecular weight of the vapor is significantly less than the one of the liquid, namely \( m_1 << m_2 \) which implies that the constant

\[ L = \left(1 - \frac{m_1}{m_2}\right) \frac{1}{m_1} > 0, \]  

(1.8)
and so the pressure law becomes
\[ p = \frac{a}{3} \theta^4 + \frac{\rho}{m_2} R \theta + LR \rho_1 \theta + p_c(\rho). \]
Motivated by the above discussion, here and in what follows, the pressure law has the general form
\[ p = p(\rho, \theta, \rho_1) = p_c(\rho) + \frac{R}{m_2} p_0(\rho) \theta + LR \rho_1 \theta + \frac{a}{3} \theta^4, \quad (1.9) \]
where \( p_0(\rho) \) is the so-called thermal pressure.

H.3 The internal energy \( e \) satisfies a general constitutive law \( e = e(\rho, \theta, f_1) \), namely
\[ e = e(\rho, \theta, f_1) = P_c(\rho) + \frac{c_v}{m_2} \theta + c_v L \frac{\rho_1 \theta}{\rho} + a \frac{\theta^4}{\rho}, \quad (1.10) \]
with \( c_v \) the specific heat and
\[ P_c(\rho) = \int_1^\rho \frac{p_e(z)}{z^2} \, dz \quad (1.11) \]
denoting the so-called elastic potential.

H.4 We treat vapor and liquid as different species, each one having each own density \( \rho_1 \) and \( \rho_2 \), pressure \( p_1 \) and \( p_2 \), molecular weights \( m_1 \) and \( m_2 \) but both components having the same temperature at each point of the mixture.

H.5 The diffusion flux \( \mathbf{F}_i \) is determined by the law
\[ \mathbf{F}_i = -d_i \nabla \log(\rho_i \theta). \quad (1.12) \]
Here \( d \) denotes the species diffusion coefficient, which is assumed to satisfy Fick’s-type law, which requires that the mixture transport coefficients and the product \( \{d \rho \} \) vary like a power of the temperature (see also [28], [27], [13]). Hence, here and in what follows, \( d \rho = D(\theta) \) denotes a function only on the absolute temperature.

In accordance with the conservation of mass,
\[ \mathbf{F}_1 + \mathbf{F}_2 = 0. \quad (1.13) \]
Remarks on constitutive relations for the diffusion flux (1.12) and their compatibility with the conservation of mass and in particular relation (1.13) can be found in [28] (see also the discussion in [36] on multicomponent flow models). In fact (1.13) comes as a result of a constraint typically imposed on certain quantities known as species thermal diffusion ratios in the diffusion flux, which for simplicity of the presentation are omitted in the above relation (we refer the reader to Ern and Giovangili [16], Weldmann and Trübenbacher [38] for further details). The underlying requirement is that the flux diffusion velocities must compensate with each other.

H.6 The heat flux \( \mathbf{q} = \mathbf{q}(\rho, \rho_i, \theta, \nabla \theta) \) is given by a very general law and consists in the present context of two parts:
\[ \mathbf{q} = \mathbf{q}_r + \mathbf{q}_f. \quad (1.14) \]
The first term \( q_F \) is determined by the *Fourier’s law*, while the second term \( q_{\text{f}_i} \) accounts for the effects of *enthalpy*, which is carried across the surface by the individual species, namely

\[
\begin{align*}
q_F &= -\kappa F(\rho, \theta) \nabla \theta, \\
q_{\text{f}_i} &= -\frac{9\gamma - 5}{4} L d \rho_i \theta \nabla \log(\rho_i \theta),
\end{align*}
\]

(1.15)

\( L \) the constant introduced in (1.8). The term \( q_{\text{f}_i} \) constitutes an additional contribution to \( q \) in binary and multicomponent systems.

As in Fan [18] Fan and Slemrod [19], and in an attempt to simplify the analysis, while at the same time preserving the basic wave patterns and the interesting phenomena of the flow observed in experiments, we take

\[
w_{\text{growth}} = K f_1(1 - f_1).
\]

(1.16)

Here \( f_1(1 - f_1) \) is the probability of collision between the particles of liquid and vapor (facilitating that way the reaction) and \( K \) is known as the *rate parameter*. In combustion theory the rate function is typically given by *Arrhenius law*. However, in liquid-vapor phase transitions the rate of phase change varies significantly according to \( |p - p_0| \), with \( p_0 \) denoting the so-called *equilibrium pressure*.

The expression of the rate constant \( K \) should be adjusted according to the following physical considerations.

(a) When \( p_0(\theta) - p(\rho, \rho_i, \theta) > 0 \), then liquid tends to evaporate and hence \( w_{\text{growth}} \geq 0 \).

(b) Similarly when \( p_0(\theta) - p(\rho, \rho_i, \theta) < 0 \), then vapor tends to condense into liquid and hence \( w_{\text{growth}} \leq 0 \).

Consistent with the above description we shall model the growth of nuclei of vapor by

\[
w_{\text{growth}} = K f_1(1 - f_1), \quad K = \frac{|p - p_0|}{bp_0},
\]

(1.17)

with \( b > 0 \) a constant denoting the typical reaction time. We remark that the term \( w_{\text{nucleation}} \) in (1.4) is typically not equal to 0 when \( p \neq p_0 \) but it is relatively small in the metastable regions, so following Fan [18], Fan and Slemrod [19] and Slemrod [34] we take

\[
w_1(\rho, f_1) = w_{\text{growth}}.
\]

(1.18)

Multiplying the *conservation of mass* equation in by \((\rho P_\varepsilon(\rho))'\) we obtain

\[
\partial_t(\rho P_\varepsilon(\rho)) + \text{div}(\rho P_\varepsilon(\rho) \mathbf{u}) + p_\varepsilon(\rho) \text{div} \mathbf{u} = 0
\]

(1.19)

and so the *conservation of energy* equation yields the *thermal equation*

\[
\partial_t \left[ a \theta^4 + \frac{c_v}{m_2} \rho \theta + L c_v \rho_1 \theta \right] + \text{div} \left[ \left( a \theta^4 + \frac{c_v}{m_2} \rho \theta + L c_v \rho_1 \theta \right) \mathbf{u} \right] \\
+ \text{div} \mathbf{q} = S : \nabla \mathbf{u} - \left[ \frac{a}{3} \theta^4 + \frac{R}{m_2} p_\varepsilon(\rho) \theta + LR \rho_1 \theta \right] \text{div} \mathbf{u}.
\]

(1.20)
The internal energy is related with the specific entropy $s$ through a rather general thermodynamic relation

$$
\theta \mathbf{D} s = \mathbf{D} e + p \mathbf{D} \left( \frac{1}{\rho} \right) - g(\rho, \rho_i, \theta) \mathbf{D} f_i, \quad (1.21)
$$

where $\mathbf{D}$ denotes the total differential, and $g$ a function, which depends on the specific physical context [28].

If the motion is smooth, starting from the energy balance equation and in accordance with (1.21) we derive now the entropy equation, which now reads,

$$
\partial_t (\rho s) + \text{div}(\rho s \mathbf{u}) + \text{div} \left[ - \frac{k(\theta)}{\theta} \nabla \theta - (\rho s f_i) L d \nabla \log(\rho_1 \theta) \right] = r.
\quad (1.22)
$$

Here $\mathbf{r}$ denotes the entropy production, which is expressed by

$$
r = \frac{1}{\theta} \left( \mathbf{S} : \nabla \mathbf{u} + \mathbf{r} \frac{\theta}{\rho} \frac{1}{\theta} + L \rho_1 \theta \mathbf{d} |\nabla \log(\rho_1 \theta)|^2 \right), \quad (1.23)
$$

and the specific entropy $s$ is given by

$$
s = s_F + s_{f_i}, \quad (1.24)
$$

where

$$
\begin{align*}
\begin{cases}
 s_F = \frac{4 \alpha_1}{3} \rho^3 + \frac{c_s}{m_2} \log(\theta) - \frac{R}{m_2} P_{b}(\rho), \\
s_{f_i} = L c_i f_i \log(\theta) - L R f_1 \log(\rho_1),
\end{cases}
\quad (1.25)
\end{align*}
$$

where

$$
P_b(\rho) = \int_1^\rho \frac{p_b(z)}{z^2} dz.
$$

The reader should contrast the form of the entropy in the present context with the form of the entropy in earlier articles (cf. [12], [13], [14], [22]) to see the effect of the presence of the individual components in the mixture.

Note that the entropy equation (1.22) is derived by dividing the thermal equation (1.20) by the absolute temperature $\theta$, with the aid of the balance laws and information for the evolution of certain quantities related to the entropy namely $\{\rho_1 \log \rho_1\}$ and $\{\rho P_b(\rho)\}$

$$
\begin{align*}
\partial_t (\rho_1 \log(\rho_1)) + \text{div}(\rho_1 \log(\rho_1) \mathbf{u}) = \text{div}(\rho_1 \mathbf{d} \log(\rho_1 \theta)) (\log(\rho_1) + 1) - \rho_1 \text{div} \mathbf{u} \\
\partial_t (\rho P_b(\rho)) + \text{div}(\rho P_b(\rho) \mathbf{u}) + p_b(\rho) \text{div} \mathbf{u} = 0
\end{align*}
\quad (1.26)
$$

and it is in agreement with the thermodynamic relation (1.21) for appropriate choice of function $g = g(\rho, \theta, \rho_i)$.

In the case of a general nonsmooth motion now, and in the spirit of the second law of thermodynamics as given in Truesdell [37], we can only assert that

$$
\partial_t [\rho (s_F + s_{f_i})] + \text{div} [\rho (s_F + s_{f_i}) \mathbf{u}] + \text{div} \left( - \frac{k(\theta)}{\theta} \nabla \theta - (\rho s f_i) L d \nabla \log(\rho_1 \theta) \right) \\
\geq \frac{1}{\theta} \left( \mathbf{S} : \nabla \mathbf{u} + k(\theta) \frac{1}{\theta} \left| \nabla \theta \right|^2 + L \rho_1 \theta d |\nabla \log(\rho_1 \theta)|^2 \right). \quad (1.27)
$$
The equations which characterize the liquid-vapor phase-transition now read

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0 \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \rho &= \text{div}\mathbf{S} \\
\partial_t (\rho \mathbf{s}) + \text{div}(\rho \mathbf{s} \mathbf{u}) + \text{div}\left[-\frac{k(\theta)}{\theta} \nabla \theta - (\rho s_f) L d \nabla \log(\rho_1 \theta)\right] &= \mathbf{r} \\
\partial_t (\rho_1) + \text{div}(\rho_1 \mathbf{u}) + \text{div}\mathcal{F}_1 &= \mathbf{w}_1.
\end{align*}
\]  

(1.28) \quad (1.29) \quad (1.30) \quad (1.31)

We assume that the mixture occupies a bounded domain \( \Omega \subset \mathbb{R}^N, N = 3 \) of class \( C^{2+\nu}, \nu > 0 \), and the whole physical system is both mechanically and thermally isolated. The following boundary conditions hold

\[
\mathbf{u}|_{\partial \Omega} = 0, \quad \mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \mathcal{F} \cdot \mathbf{n}|_{\partial \Omega} = 0,
\]  

(1.32)

where \( \mathbf{n} \) denotes the outer normal vector to \( \partial \Omega \).

In accordance with the above discussion we infer that the total energy

\[
E = \frac{1}{2} \rho|\mathbf{u}|^2 + \rho e + h \rho f_1
\]  

(1.33)

is constant of motion, specifically,

\[
\frac{d}{dt} \int_{\Omega} E(t) dx = 0.
\]  

(1.34)

We consider the following initial conditions:

\[
\begin{cases}
\rho(0,\cdot) = \rho_0, \\
(\rho \mathbf{u})(0,\cdot) = \mathbf{m}_0, \\
(\rho \mathbf{s})(0,\cdot) = \rho_0 s_0, \\
(\rho f_1)(0,\cdot) = \rho_0 f_{1a},
\end{cases}
\]  

(1.35)

together with the compatibility condition:

\[
\mathbf{m}_0 = 0 \quad \text{on the set} \quad \{x \in \Omega | \rho_0(x) = 0\}.
\]  

(1.36)

The objective of this work is to establish the global existence of weak solutions to this initial boundary value problem with large initial data. This article extends earlier work on phase transitions (cf. Donatelli and Trivisa [12] [13]) since it now takes into consideration both the species concentration as well as the unique character of the individual components as given by their distinct molecular weights. The constitutive relations presented here differ from the ones given in [12], [13], are able to accommodate both binary and multicomponent fluid mixtures, while at the same time yield positive entropy production in the spirit of the second law of thermodynamics as presented in Truesdell [37].

The general character of the constitutive relations imposes new mathematical difficulties, namely

- Obtaining boundness of the oscillation defect measure (essential for the strong convergence of the density \( \rho \)) requires the special treatment of additional terms in the pressure and the equations.
- The presence of extra terms in the entropy production can be handled only by deriving new and rather delicate energy and entropy estimates.

We consider the following initial conditions:
The pressure estimates need refinement in dealing with concentration phenomena.

The methods of use are weak convergence methods and compactness arguments in the spirit of Feireisl [22] and Lions [32]. In the heart of the analysis lies the quantity

\[ p - \left( \frac{4}{3} \mu + \zeta \right) \text{div} u, \]

known as the effective viscous pressure. The weak continuity property of the effective viscous pressure was first shown by P.L. Lions [32] for the barotropic Navier-Stokes equations, where \( p = p(\rho) \) and with constant viscosity coefficients. This result was extended to include the case of general temperature dependent viscosity coefficients by Feireisl [23] with the aid of delicate commutator estimates in the spirit of Coifman and Meyer [8].

In Section 4 we establish the weak continuity of the effective viscous pressure, which involves showing that the following relation holds true

\[
\left( p(\rho, \theta, f_1) - \left( \frac{4}{3} \mu(\theta) + \zeta(\theta) \right) \text{div} u \right) b(\rho) = \\
\left( p(\rho, \theta, f_1) b(\rho) - \frac{4}{3} (\mu(\theta) + \zeta(\theta)) \text{div} u \right) \bar{b}(\rho)
\]

for any bounded function \( b \).

In the center of the analysis lies also the requirement that \( \rho \) is a renormalized solution of the continuity equation, a notion introduced by DiPerna and Lions [10], [11]. The role of this concept is two-fold, in the sense that it helps us deal both with the problem of density oscillations as well as with the problem of temperature concentration phenomena.

Another key ingredient in evaluating the propagation of density oscillations is in fact showing boundness of a suitable oscillation defect measure, which is typically expressed in terms of certain cut-off functions (cf. Feireisl [22]). We remark that the choice of the constitutive relations effects in a crucial way the form of this measure.

The outline of this article is as follows. In Section 2 and 3 we present the weak formulation of the problem and state the main result. Our analysis relies on the concept of variational solution, which provides us with the appropriate space setting for the admissible solutions of our system.

In Section 4 we introduce a series of approximating problems constituting a three level approximating scheme. This scheme consists of a set regularized equations (see also [12], [13], [22]). The regularization appears in terms of additional \( \varepsilon \) and \( \delta \) quantities accounting for artificial viscosity and artificial pressure. In Section 5 we obtain uniform bounds (energy and entropy estimates) necessary as we let the artificial viscosity \( \varepsilon \) go to zero in Section 6. Here, the weak continuity of the effective viscous pressure needs to be established in order to obtain suitable estimates on the density component \( \rho \).

In Section 7 we recover the original system by letting \( \delta \) go to zero getting rid of the artificial pressure term \( \{\delta \rho^3\} \). This process requires the introduction of suitable family of functions

\[ T_k(z) = kT \left( \frac{z}{k} \right) \text{ for } z \in \mathbb{R}, k = 1, 2, \ldots \]
where \( T \in C^\infty \) is a suitable cut-off function, whose choice depends in a crucial way on the particular physical context, namely on the choice of constitutive relations for the equation of state for the pressure, and other quantities in the system as well as the assumptions on the viscosity coefficients. Of course, the main goal here is to show that the so-called oscillation defect measure

\[
osc_{\beta+1}\left[ \rho \psi \to \rho \right]((0,T) \times \Omega) = \sup_{k \geq 1} \left( \limsup_{\delta \to 0} \int_0^T \int_{\Omega} \left| T_k(\rho \delta) - T_k(\rho) \right|^{\beta+1} \, dx \, dt \right)
\]

is bounded.

Related results on phase-transition models are presented in a series articles by Donatelli and Trivisa [12], [13] in the context of combustion models and by Feireisl et al [14], [22], [23], [27] for related models in fluids and astrophysics. We refer the reader also to the discussion in [36], where a wide class of multicomponent models and their physical properties are investigated.

The setting of the present article is motivated by physical considerations presented in a series of articles by Fan [18], Fan and Slemrod [19] and Slemrod [34]. For related work on van der Waals fluids and the issue of non-uniqueness of phase transition near the Maxwell line we refer the reader to Benzoni-Gavage [2] and the references therein, while for general discussion on nucleation phenomena we refer the reader to Springer [35].

2. Weak Formulation. Our objective in this article is the solvability of the initial-boundary value problem (1.28)-(1.31) together with (1.32) and (1.35) for large initial data. To this end, we need to rely on the concept of variational solution. Philosophically this weak formulation is connected to the balance laws of continuum physics. Balance laws are typically expressed in the form of integral identities and therefore no requirement on the regularity of the integrand quantities is imposed.

2.1. Dissipation of mechanical energy. In the framework of weak solutions, it is common to replace the (formally derived) classical entropy equality by an inequality (cf. Truesdell [37]). This is due to the possible loss of some part of the kinetic energy of the system. This loss of energy appears mathematically in the form of a measure. In the physical context this portion of the energy may be regarded as a new part of the spatial domain. For further remarks on the topic, we refer the reader to Feireisl [22] for relevant discussion in the context of compressible fluids, as well as to Dafermos [9] for relevant discussion in the context of hyperbolic conservation laws.

The variational formulation of the entropy production is given by

\[
\int_0^T \int_{\Omega} \left\{ (\rho s) \partial_t \phi + (\rho s) \mathbf{u} \cdot \nabla \phi + \left[ -\frac{k(\theta)}{\theta} \nabla \theta - (\rho s f_1) LD(\theta) \nabla \log(\rho_1 \theta) \right] \cdot \nabla \phi \right\} \, dx \, dt \\
\leq \int_0^T \int_{\Omega} \left\{ \frac{\mathbf{S} : \nabla \mathbf{u}}{\theta} - \frac{k(\theta)}{\theta^2} \left( \frac{\partial \theta}{\partial t} \right)^2 - L f_1 D(\theta) \left| \nabla \log(\rho_1 \theta) \right|^2 \right\} \phi \, dx \, dt, \quad (2.1)
\]

for any nonnegative function \( \phi \in \mathcal{D}((0,T) \times \mathbb{R}^N) \).

Note that the fact that \( \theta \) appears in the denominator in the above relation \( \frac{\mathbf{S} : \nabla \mathbf{u}}{\theta} \) indicates that the absolute temperature \( \theta \) must be positive in order for (2.1) to be meaningful. In addition the presence of term \( \{ f_1 D(\theta) \left| \nabla \log(\rho_1 \theta) \right|^2 \} \) indicates that if the density fraction \( f_1 \) is strictly positive then the density of the vapor \( \rho_1 \) must be positive in order for (2.1) to be meaningful. This issue will be discussed in the sequel.
Motivated by this discussion, we introduce now the notion of a \textit{variational solution} to the initial boundary value problem (1.28)-(1.31) together with (1.32), (1.35) and (1.36).

2.2. The class of admissible solutions. In this section we present the class of admissible solutions for our system (1.28)-(1.31) motivated by the underlying physical principles of continuum physics.

**Definition 2.1.** We say that \((\rho, u, \theta, f_1)\) is a variational solution of the initial boundary value problem (1.28)-(1.31) on the interval \((0, T)\) if it satisfies the following properties:

(a) **Balance of mass:**
The density \(\rho\) is a nonnegative function,
\[
\rho \in C([0, T]; L^1(\Omega)) \cap L^\infty(0, T; L^1(\Omega)), \quad \rho(0, \cdot) = \rho_0
\]
satisfying the integral identity:
\[
\int_0^T \int_\Omega \rho \partial_t \psi + \rho \mathbf{u} \cdot \nabla \psi \, dx \, dt = 0, \tag{2.2}
\]
for any \(\psi \in C^\infty([0, T] \times \bar{\Omega}), \quad \psi(0) = \psi(T) = 0.

In addition, we require that \(\rho\) is a “renormalized solution” of (1.28), in the sense that the following integral identity is also verified
\[
\int_0^T \int_\Omega b(\rho) \partial_t \psi + b(\rho) \mathbf{u} \cdot \nabla \psi + (b(\rho) - b'(\rho) \rho) \text{div} \mathbf{u} \psi \, dx \, dt = 0, \tag{2.3}
\]
for any \(b \in C^1[0, \infty)\) such that \(b'(\rho) = 0\) for all \(\rho\) large enough, and any test function
\[
\psi \in C^\infty([0, T] \times \bar{\Omega}), \quad \psi(0) = \psi(T) = 0.
\]

(b) **Balance of momentum:**
The velocity \(\mathbf{u}\) belongs to the class
\[
\mathbf{u} \in L^a(0, T; W^{1,b}_0(\Omega)), \quad b > 1, \quad \rho \mathbf{u}(0, \cdot) = \mathbf{m}_0,
\]
and the momentum equation (1.2) holds in \(D'((0, T) \times \Omega)\) in the sense that
\[
\int_0^T \int_\Omega \rho \mathbf{u} \partial_t \psi + \rho (\mathbf{u} \otimes \mathbf{u}) : \nabla \psi + p \text{div} \psi \, dx \, dt = \int_0^T \int_\Omega S : \nabla \psi \, dx \, dt, \tag{2.4}
\]
for all \(\psi \in [D((0, T) \times \Omega)]^3\).

(c) **Balance of energy:**
The temperature \(\theta\) is a nonnegative function,
\[
\theta, \log(\theta) \in L^2(0, T; W^{1,2}(\Omega)).
\]
The specific entropy \(s\), determined in terms of other state variables as described in (1.24), (1.25) as well as the terms in (2.1) are integrable and the integral relation (2.1) holds for any nonnegative function \(\phi \in D((0, T) \times \mathbb{R}^3)\).
Moreover, as there is no flux of energy through the kinematic boundary we require the total energy $E(t)$ to be constant of motion, that is
\[
\int_0^T E(t) \partial_t \psi \, dt = 0, \quad \text{for any } \psi \in \mathcal{D}((0,T)).
\] (2.5)

(d) **Balance of species mass:**
The volume fraction $f_1$ belongs in the class
\[
f_1 \in L^2(0,T; W^{1,2}(\Omega)), \quad \rho_1 = \rho f_1
\]
and
\[
0 \leq f_1(x,t) \leq 1, \quad \text{for a.a. } (x,t) \in (0,T) \times \Omega
\]
and the kinetic equation of species holds in $\mathcal{D}'((0,T) \times \Omega)$
\[
\int_0^T \int_\Omega \rho_1 \partial_t \psi + \rho_1 u \cdot \nabla \psi \, dx \, dt = \int_0^T \int_\Omega w_1 \partial_t \psi + D(\theta) \nabla \log(\rho_1 \theta) \cdot \nabla \psi \, dx \, dt,
\] (2.6)
for all $\psi \in [\mathcal{D}((0,T) \times \Omega)]^N$.

(e) The functions $\rho, \rho u, \rho \theta$ and $\rho f_1$ satisfy the initial conditions in the weak sense.

2.3. **Assumptions.**

**Pressure**
The pressure $p$ obeys the general pressure law (1.9) where the elastic pressure $p_e$ and the thermal pressure $p_\theta$ are continuously differentiable function of the density and in addition satisfy the properties (see also [22], [12], [13])
\[
\begin{align*}
&\begin{cases}
p_e(0) = 0, \\
p_e'(\rho) \geq a_1 \rho^{\gamma - 1} - c_1, \quad a_1 > 0
\end{cases} \\
p_e(\rho) \leq a_2 \rho^{\gamma} + c_2
\end{align*}
\] (2.7)
with
\[
\gamma \geq 2, \quad \gamma > \frac{4G}{3}
\]

**Transport coefficients**

- The viscosity parameters depend on the absolute temperature in the following fashion.
\[
\begin{align*}
0 < \mu(1 + \theta^\alpha) &\leq \mu(\theta) \leq \bar{\mu}(1 + \theta^\alpha), \\
0 < \xi \theta^\alpha &\leq \zeta(\theta) \leq \bar{\zeta}(1 + \theta^\alpha)
\end{align*}
\] (2.8)
for $\alpha \geq \frac{1}{2}$.
- The heat conductivity obey the rule
\[
\begin{align*}
k = k_C(\theta) + \sigma \theta^3, \\
0 < k_C &\leq k_C(\theta) \leq \bar{k}_C(1 + \theta^3)
\end{align*}
\] (2.9)
where the term $\{\sigma \theta^3\}$ with $\sigma > 0$ accounts for the radiative effects.
• The species diffusion coefficient $D$ is a continuously differentiable function depending only on the absolute temperature in the following way

$$0 < D < D(\theta) \leq \tilde{D}(1 + \theta^3)$$

for all $\theta > 0$.

3. Main results. We are now ready to state the existence result for the initial boundary value problem introduced in Section 1.

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $\partial \Omega \in C^{2+\nu}, \nu > 0$. Suppose that the pressure $p$ is determined by the equation of state (1.9), with $a > 0$, and $p_\infty, p_0$ satisfying (2.7). In addition, let the viscous stress tensor $\mathcal{S}$ be given by (1.5), where the viscosity coefficients $\mu$ and $\zeta$ are continuous differentiable globally Lipschitz functions of $\theta$ satisfying (2.8) for $\frac{1}{2} \leq \alpha \leq 1$. Similarly, let the heat flux $\mathbf{q}$ be given by (1.14)-(1.15) with the heat conductivity $k$ satisfying (2.9). Finally, assume that the initial data $\rho_0, \mathbf{m}_0, \theta_0, f_{10}$ satisfy

$$\begin{cases}
\rho_0 \geq 0, \rho_0 \in L^2(\Omega), \\
\mathbf{m}_0 \in [L^1(\Omega)]^3, \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega), \\
\theta_0 \in L^\infty(\Omega), 0 < \beta_0 \leq \theta_0(x) \leq \bar{\theta} \text{ for a.e. } x \in \Omega, \\
f_{10} \in L^\infty(\Omega), 0 \leq f_{10} \leq \tilde{f}_{10} \text{ a.e. in } \Omega, \quad \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega).
\end{cases} \quad (3.1)$$

Then, for any given $T > 0$ the initial boundary value problem (1.28)-(1.31), with (1.32), (1.35) the boundary and initial conditions respectively, has a variational solution on $(0, T) \times \Omega$. More specifically, there exist functions $\rho, \mathbf{u}, \theta, f_1$ satisfying the following properties:

• The density $\rho$ is a nonnegative function,

$$\rho \in C([0, T]; L^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \rho(0, \cdot) = \rho_0.$$

The momentum $\rho \mathbf{u}$

$$\rho \mathbf{u} \in C([0, T]; L^{\frac{2\alpha}{\alpha+1}}(\Omega; \mathbb{R}^3))$$

and the integral identities (2.2), (2.3) hold for any $\psi \in [D((0, T) \times \mathbb{R}^3)]$.

• The velocity $\mathbf{u}$ belongs to the class

$$\mathbf{u} \in L^a(0, T; W^{1,b}_0(\Omega)), b > 1, \quad \rho \mathbf{u}(0, \cdot) = \mathbf{m}_0;$$

the quantities $\rho \mathbf{u} \otimes \mathbf{u}, p, \mathcal{S}$ are integrable on $(0, T) \times \Omega$, and the integral identity (2.3) holds for any $\psi \in [D((0, T) \times \mathbb{R}^3)]$.

• The temperature $\theta$ is a nonnegative function,

$$\theta, \log(\theta) \in L^2(0, T; W^{1,2}(\Omega)).$$

The entropy $s$ as well as the terms in (2.1) are integrable on $(0, T) \times \Omega$ and the inequality (2.1) holds for any nonnegative function $\phi \in D((0, T) \times \mathbb{R}^3)$.

Moreover,

$$\text{ess lim}_{t \to 0^+} \int_\Omega \rho s(t) \phi \, dx \geq \int_\Omega \rho_0 s_0 \phi \, dx, \text{ for any nonnegative } \phi \in D(\Omega),$$
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where
\[
\rho_0s_0 = \frac{4a_3}{3} \theta_0^3 + \frac{c_v}{m_2} \rho_0 \log(\theta_0) - \frac{R}{m_2} \rho_0 P_0(\rho_0) + Lc_v \rho_0 \log(\theta_0) - LR \rho_0 \log(\rho_0).
\]

Moreover, the total energy \(E(t)\) is constant of motion, and the integral relation (1.34) holds for any \(\psi \in \mathcal{D}(0,T)\).

- The mass fraction \(f_1 \in L^2(0,T;W^{1,2}(\Omega))\), with
\[
0 \leq f_1(x,t) \leq 1, \quad \text{for a.a. } (x,t) \in (0,T) \times \Omega
\]
and the integral equation (2.6) holds in \(\mathcal{D}'((0,T) \times \mathbb{R}^3)\).

- The functions \(\rho, \rho u, \rho \theta\) and \(\rho f_1\) satisfy the initial conditions (1.35) in the weak sense.

**Proof:** In the remaining part of this paper we will carry out the strategy outlined in the introduction. The proof of this theorem will be given in detail in Sections 4 – 7.

### 4. Solvability of Approximating Problems

In this section we construct a sequence of approximate problems by adding appropriate regularizations in the equations (1.28)-(1.31). Starting from the modified continuity equation the approximate problems now read:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) & = \varepsilon \Delta \rho \\
\nabla \cdot \mathbf{n} |_{\partial \Omega} & = 0 \\
\rho(0,\cdot) & = \rho_{0,\delta} 
\end{align*}
\]

(4.1)

For the initial approximation of the density \(\rho_{0,\delta} \in C^{2+\nu}(\Omega)\) we have also

\[
\begin{align*}
0 < \delta & \leq \rho_{0,\delta} \leq \delta^{-\frac{\nu}{2}} & \text{on } \Omega, \\
\rho_{0,\delta} & \to \rho_0 \text{ in } L^\gamma(\Omega), & |\{\rho_{0,\delta} < \rho_0\}| \to 0 \text{ for } \delta \to 0.
\end{align*}
\]

The modified momentum equation is given by

\[
\begin{align*}
\partial_t (\rho u) + \nabla (\rho u \otimes u) + \nabla (\rho \theta, f_1) + \delta \rho^2 \nabla u \cdot \nabla \rho & = \text{div} \mathbf{S}, \\
\mathbf{u} |_{\partial \Omega} & = 0, \\
\rho u(0,\cdot) & = \mathbf{m}_{0,\delta},
\end{align*}
\]

(4.2)

Moreover, the initial momenta are given by

\[
\mathbf{m}_{0,\delta}(x) = \begin{cases} 
\mathbf{m}_0 & \text{if } \rho_{0,\delta}(x) \geq \rho_0(x), \\
0 & \text{for } \rho_{0,\delta}(x) < \rho_0(x).
\end{cases}
\]

The modified thermal equation is given by:

\[
\begin{align*}
\partial_t \left[ a \theta^4 + \frac{a}{m_2} \rho \theta + RL \rho_1 \theta \right] + \text{div} \left[ \left( a \theta^4 + \frac{a}{m_2} \rho \theta + L \rho_1 \theta \right) u \right] \\
- \text{div} \left[ \left( \kappa C(\rho, \theta) + \sigma \theta^2 \right) \nabla \theta + \frac{a}{2} \rho_0 L D(\theta) f_1 \theta \nabla \log(\rho, \theta) \right] \\
= \varepsilon |\nabla \rho|^2 \left[ \frac{E_C(\rho)}{\rho} + \delta \beta \rho^2 \theta^2 \right] + \mathbf{S} : \nabla u - \left[ a \theta^4 + \frac{a}{m_2} \rho \theta + L \rho_1 \theta \right] \text{div} \mathbf{u}
\end{align*}
\]

(4.3)

\[
\begin{align*}
\nabla \theta \cdot \mathbf{n} |_{\partial \Omega} & = 0 \\
\theta(0,\cdot) & = \theta_{0,\delta},
\end{align*}
\]
The functions \( \theta_{0,\delta} \in C^{2+\nu}(\Omega) \) satisfy in addition
\[
\begin{aligned}
\nabla \theta_{0,\delta} \cdot n_{\partial \Omega} &= 0, \quad 0 < \bar{\theta} < \theta_{0,\delta} \leq \bar{\theta} \quad \text{on} \quad \Omega, \\
\theta_{0,\delta} &\to \theta_0 \quad \text{in} \quad L^1(\Omega) \quad \delta \to 0.
\end{aligned}
\]

The modified species conservation equation is expressed by
\[
\left\{
\begin{aligned}
\partial_t \rho_1 + \text{div}(\rho_1 u) + \varepsilon \nabla f_1 \cdot \nabla \rho &= K f_1 (f_1 - 1) + \text{div}(f_1 D(\theta) \nabla \log(\rho_1 \theta)), \\
\nabla f_1 \cdot n_{\partial \Omega} &= 0 \\
f_1(0, \cdot) &= f_{1_0}
\end{aligned}
\right.
\] (4.4)

Finally, the initial approximations of the mass fraction of the reactant \( f_{1_0,\delta} \in C^{2+\nu}(\Omega) \) satisfy
\[
\begin{aligned}
\nabla f_{1_0,\delta} \cdot n_{\partial \Omega} &= 0, \quad 0 \leq f_{1_0,\delta} \leq 1 \quad \text{on} \quad \Omega, \\
f_{1_0,\delta} &\to f_{1_0} \quad \text{in} \quad L^1(\Omega) \quad \delta \to 0.
\end{aligned}
\]

Next, we decompose the elastic pressure component \( p_e(\rho) \) of the pressure in two parts \( p_m \) and \( p_b \)
\[
p_e(\rho) = p_m(\rho) + p_b(\rho),
\] (4.5)

where the former \( p_m \) is a non-decreasing function, the latter \( p_b \) is a bounded function on \([0, \infty)\), while both are elements in \( C([0, \infty)) \) (see also [14], [13]). The reason for this decomposition will be apparent in the sequel, where the properties of the two components will be instrumental in obtaining useful energy and entropy estimates.

We remark that an important consideration at this point is the necessity to keep the total energy constant at each step of the approximation. The addition of the artificial pressure \( \rho \)-terms, on the other hand is essential in order to ensure that the pressure estimates hold true even as the artificial viscosity vanishes, and in resolving some technical issues related to temperature estimates. Note that the parabolic regularization of the continuity equation yields, at this level of the approximation, lower and upper bound for the density \( \rho_0 \).

The sequence of approximate solutions \( \{\rho_n, u_n, \theta_n, f_{1_n}\} \) for the system (4.1)-(4.4) for fixed \( \varepsilon, \delta \) can now be constructed in a variety of ways, say for instance using the Faedo Galerkin method, which involves replacing the regularized balance of momentum equation by a set of integral relations, with \( \rho, \theta \) and \( f_1 \) being exact solutions of (4.1), (4.3) and (4.4).

The approximate velocities \( u_n \in C([0, T]; X_n) \) satisfy a set of integral equations of the form
\[
\int_\Omega \rho u_n(\tau) \cdot \eta \, dx - \int_\Omega m_{0,\delta} \cdot \eta = \\
\int_0^\tau \int_\Omega \left( \rho u_n \otimes u_n - S_n \right) : \nabla \eta + \left( p_m(\rho) + \frac{\alpha}{3} \theta^3 + \frac{R}{m_2} \theta \rho_0(\rho) + L \rho \theta + \delta \rho \right) \text{div} \eta \, dx \, dt
\]
\[
+ \int_0^\tau \int_\Omega p_b(\rho) \text{div} \eta - \varepsilon \nabla u_n \nabla \rho \eta \, dx \, dt,
\] (4.6)

for any test function \( \eta \in X_n = \text{span}\{\eta_j\}_{j=1}^n \), \( \eta_j \in D(\Omega)^N \) and all \( \tau \in [0, T] \).

Our approach is in the spirit of Feireisl [22] and involves:
Solving the approximate problems (4.1)-(4.4) for fixed $\varepsilon$ and $\delta$.

Passing to the limit for $\varepsilon \to 0$.

Passing to the limit for $\delta \to 0$.

For the solvability of the approximate problems few remarks are in order.

- The solvability of the regularized continuity equation (4.1) is obtained by a simple fixed point argument (see [22]). The solvability of the regularized momentum equation is given by the Faedo-Galerkin method, with $\rho, \theta, f_1$ are obtained from (4.1), (4.3), (4.4).

- The modified thermal equation (4.3) is a non-degenerate parabolic equation with respect to $U = \theta^4$ with sublinear coefficients, hence the existence of the solution temperature $\theta_n$ follows easily (we refer the reader also to the articles [12], [13], [14], [31] where relevant equations were treated).

- The regularized species conservation equation (4.4) is a parabolic quasi-linear equation. Applying standard techniques [17], [31] one can deduce the existence and uniqueness of a solution (see [12], [13]). The only delicate issue here is the fact that the coefficients in (4.4) are not regular in time, therefore some special regularization in time needs to be introduced in the spirit of [12] (see also [14], [22]).

5. Uniform Bounds.

5.1. Energy Estimates. As stated in the Definition 2.1 the total energy is constant of motion. Boundness of the total energy gives rise to the following estimates:

$$\sqrt{\rho} |u| \text{ bounded in } L^\infty(0, T; L^2(\Omega))$$  \hfill (5.1)

$$\rho \text{ bounded in } L^\infty(0, T; L^2(\Omega))$$  \hfill (5.2)

$$\rho \varepsilon \text{ bounded in } L^\infty(0, T; L^1(\Omega))$$  \hfill (5.3)

$$\rho f_1 \text{ bounded in } L^\infty(0, T; L^1(\Omega))$$  \hfill (5.4)

and

$$a \, \text{esssup}_{t \in (0, T)} \int_\Omega \theta^4(t) \, dx \leq c.$$

5.2. Entropy estimates. The existence of the approximate solution sequence $\{\rho_n, u_n, \theta_n, f_{1n}\}$ has now been established and we are ready to proceed in the program outlined in Section 4.

Considering a suitable choice of test functions $\eta = u_n(t)$ in (4.6), integrating in space both the (regularized) thermal equation (4.3) and the species conservation equation (4.4) and adding the resulting relations gives rise to an energy inequality of the form

$$\frac{d}{dt} \int_\Omega \left[ \frac{1}{2} \rho_n |u_n|^2 + \rho_n P_m(\rho_n) + \frac{\delta}{\beta-1} \rho_n^{\beta} + \rho_n f_{1n} + a \theta_n^4 + \frac{c_v}{m_2} \theta_n \rho_n + RL \rho_1 \theta \right] dx$$

$$= \int_\Omega \rho_n \text{div} u_n dx + \int_\Omega f_{1n}(f_{1n} - 1) dx - \varepsilon \int_\Omega \nabla \rho_n \nabla f_{1n} dx,$$  \hfill (5.5)

with

$$P_m(\rho) = \int_1^\rho \frac{p_m(z)}{z^2} dz.$$
An application of the maximum principle yields that the temperature \( \theta_n \) is strictly positive, therefore we can express the equation (4.3) as an entropy inequality.

\[
\partial_t (\rho_n s_n) + \text{div}(\rho_n s_n \mathbf{u}_n) + \text{div} \left( \frac{-\kappa C(\theta_n) + \sigma \theta_n^3}{\theta_n} \nabla \theta_n - s_{f_1} L D(\theta_n) \nabla \log(\rho_1, \theta_n) \right)
\geq \frac{1}{\theta_n} \left( \mathbf{S} : \nabla \mathbf{u}_n + \frac{\kappa C(\theta_n) + \sigma \theta_n^3}{\theta_n^2} |\nabla \theta_n|^2 + L f_{1n} \theta_n D(\theta) |\nabla \log(\rho_1, \theta_n)|^2 \right)
\]
\[+ c_v \frac{m_2}{m_2} (\log(\theta_n) - 1) (\nabla \rho_n + K f_{1n} (f_{1n} - 1) - \varepsilon \nabla f_{1n} \nabla \rho_{1n}) \tag{5.6}
\]

The regularized continuity equation (4.1) multiplied by \( \rho_n \) and integrated over \( \Omega \) yields:

\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho_n^2 dx + \varepsilon \int_{\Omega} |\nabla \rho_n|^2 dx = -\frac{1}{2} \int_{\Omega} \rho_n^2 \text{div} \mathbf{u}_n dx \tag{5.7}
\]

Now (5.5), (5.6), (5.7) give rise to

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho_n |\mathbf{u}_n|^2 + \rho_n P_n(\rho_n) + \frac{\delta}{\beta - 1} \rho_n^\beta + a \theta_n^4 + c_v \frac{m_2}{m_2} \rho_n \theta_n + RL \rho_{1n} \theta \right) dx +
\]
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho_n^2 - \rho_n (s_{f_1} + s_{f_2}) + \rho_n f_{1n} + \varepsilon |\nabla \rho_n|^2 + \varepsilon |\nabla \rho_n| f_{1n} \right) dx +
\]
\[
\int_{\Omega} \left( \mathbf{S} : \nabla \mathbf{u}_n + \frac{\kappa C(\theta_n) + \sigma \theta_n^3}{\theta_n^2} |\nabla \theta_n|^2 + K f_{1n} D(\theta_n) |\nabla \log(\rho_1, \theta_n)|^2 \right) dx
\]
\[
\leq \int_{\Omega} \left( p_0(\rho_n) - \frac{1}{2} \rho_n^2 \right) \text{div} \mathbf{u}_n dx + \int_{\Omega} \frac{c_v}{m_2} (\log(\theta_n) - 1) K f_{1n} (f_{1n} - 1) dx
\]
\[+ \varepsilon \int_{\Omega} \frac{c_v}{m_2} (\nabla \log(\theta_n) \nabla \rho_n + (\log(\theta_n) - 1) \nabla f_{1n} \nabla \rho_n) dx \tag{5.8}
\]

**5.3. Dissipation Estimates.** Starting from the Newton’s law for viscosity and using the hypotheses (2.8) on viscosity coefficients and Hölder’s inequality we deduce (see also [14], [13]) that

\[
|\nabla \mathbf{u}_n + \nabla \mathbf{u}_n^{\ell} |^b c (\theta_n^{-1} |\nabla \mathbf{u}_n + \nabla \mathbf{u}_n^{\ell} |^2 + \theta_n^4) \quad \text{where} \quad b = \frac{8}{5 - \alpha}. \tag{5.9}
\]

Moreover, the hypotheses (2.9) on the heat conductivity gives us the following bounds on the absolute temperature

\[
\int_{\Omega} |\nabla \log(\theta_n)|^2 + |\nabla \theta_n|^2 dx \leq \int_{\Omega} \frac{\kappa C(\theta_n) + \sigma \theta_n^3}{\theta_n^2} |\nabla \theta_n|^2 dx \tag{5.10}
\]

Now taking into consideration (5.8), (5.9) and (5.10) we get the following estimates

\[
\sup_{t \in [0, T]} \left\{ \|\rho_n\|_{L^\infty(\Omega)} + \|\rho_n \mathbf{u}_n\|^2_{L^1(\Omega)} + \|\rho_n \theta_n\|_{L^1(\Omega)} + \|\rho_{1n}\|_{L^1(\Omega)} \right\} \leq c(\delta) \tag{5.11}
\]

\[
\sup_{t \in [0, T]} \left\{ \|\log(\theta_n)\|_{L^1(\Omega)} + \|\rho_n \log(\theta_n)\|_{L^1(\Omega)} + \|\theta_n\|_{L^1(\Omega)} \right\} \leq c(\delta) \tag{5.12}
\]

\[
\sup_{t \in [0, T]} \left\{ \|\rho_{1n} \log(\theta_n)\|_{L^1(\Omega)} + \|\rho_{1n} \theta_n\|_{L^1(\Omega)} + \|\rho_{1n} \log(\rho_{1n})\|_{L^1(\Omega)} \right\} \leq c(\delta) \tag{5.13}
\]

\[
\int_0^T \int_{\Omega} \frac{\mathbf{S} : \nabla \mathbf{u}_n}{\theta_n} + |\nabla \log(\theta_n)|^2 + |\nabla \theta_n|^2 + \varepsilon |\nabla \rho_n|^2 + |f_{1n} | \nabla \log(\rho_{1n}, \theta_n)|^2 |dx dt \leq c(\delta) \tag{5.14}
\]
and
\[ \|u_n\|_{L^b([0,T;W^{1,b}_0(\Omega)])} \leq c(\delta) \quad \text{with} \quad b = \frac{8}{5 - \alpha}. \] (5.15)

The first level of approximate solutions are constructed as a limit of \( \rho_n, u_n, \theta_n \) and \( f_n \) for \( n \to \infty \). Starting from the continuity equation and taking into consideration standard compactness arguments by the aid of (5.11)-(5.15) we get
\[ \rho_n \to \rho \quad \text{in} \quad C([0,T], L^\beta_{\text{weak}}(\Omega)). \]

By using the estimates obtained in the previous steps we can assume
\[ u_n \rightharpoonup u \quad \text{weakly in} \quad L^b([0,T;W^{1,b}_0(\Omega)]), \]
\[ \rho_n u_n \rightharpoonup \rho u \quad \ast\text{-weakly in} \quad L^\infty([0,T;L^{2\alpha}(\Omega))], \]
where \( \rho, u \) satisfy equation ((4.1)) together in the sense of distribution.

5.4. Gradient density estimate. Using standard \( L^p - L^q \) estimates for the regularized continuity equation (4.1) and taking into consideration (5.11)-(5.15) we get in the spirit of Feireisl [22]
\[ \varepsilon^n \int_0^T \|\nabla \rho_n\|_{L^{2\alpha}(\Omega;\mathbb{R}^3)} \, dt \leq c(\delta, p) \quad \text{for any} \quad p \in [1, \infty). \]

Using now the Sobolev embedding theorem \( W^{1,r}_0(\Omega) \subset L^{2\alpha}(\Omega) \) and standard interpolation estimates we get
\[ \varepsilon\|\nabla \rho_n\|_{L^q([0,T;L^n(\Omega;\mathbb{R}^3)]}) \quad \text{for some} \quad q > r', \quad \text{with} \quad r = \frac{8}{5 - \alpha}. \]

In fact, one can improve the estimates on \( \rho_n \) at this point using the \( L^p \)-theory of parabolic equations to obtain:

**Lemma 5.1.** There exists \( p > 1 \) such that
\[ \partial_t \rho_n, \Delta \rho_n \quad \text{are bounded in} \quad L^p((0,T) \times \Omega), \quad p > 1, \]
indpendently of \( n \). Consequently, the limit functions \( \rho, u \) satisfy (4.1) a.e on \( (0,T) \times \Omega \) whereas the boundary condition and initial condition hold in the sense of traces.

**Proof:** The result is obtained by applying the \( L^p \)-theory of parabolic equations. This process is rather standard, we included above some of the basic steps for completeness and we refer the reader to the articles [22], [24], [14], [31].

The following lemma will be useful in order to obtain the pointwise convergence of the temperature sequence \( \theta_n \).

**Lemma 5.2.** Let \( \Omega \subset \mathbb{R}^N, \quad N \geq 2 \) be a bounded Lipschitz domain and \( \Lambda \geq 1 \) a given constant. Let \( \rho \geq 0 \) be a measurable function satisfying
\[ 0 < M \leq \int_\Omega \rho \, dx, \quad \int_\Omega \rho^\beta \leq K \quad \text{for} \quad \beta > \frac{2N}{N + 2}. \]

Then there exists a constant \( c = c(M, K) \) such that
\[ \|v\|_{L^2(\Omega)} \leq c(M, K) \left( \|\nabla v\|_{L^2(\Omega)} + \left( \int_\Omega \rho |v|^\Lambda \right)^{\frac{1}{\Lambda}} \right). \]
for any \( v \in W^{1,2}(\Omega) \).

**Proof:** For the proof we refer the reader to Lemma 5.1 in [14].

Using Lemma 5.2 and the estimates in (5.11)-(5.14) it is possible to extract a subsequence of \( \theta_n \) such that

\[
\begin{align*}
\theta_n & \to \theta \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)), \\
\theta_n & \to \theta \text{ weakly-* in } L^\infty(0, T; L^4(\Omega)), \\
\log(\theta_n) & \to \log(\theta) \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)).
\end{align*}
\]

Next, we obtain some estimates on the densities fractions \( f_i \) and the densities of the individual species \( i \):

In the context of the approximate (regularized) problems all quantities are smooth therefore an application of maximum principle in (4.4) yields uniform bounds for the density volume fraction \( f_1 \).

Multiplying now (4.4) by \( f_1 \) and integrating in space, we deduce with the aid of the estimates (5.11)-(5.15) and Gronwall’s lemma, bounds for the gradient of \( f_1 \).

These results are given in the following relation:

\[
\begin{align*}
\frac{d}{dt} \int_\Omega \frac{1}{2} \rho_n f_1^n dx + \int_\Omega D(\theta_n) |\nabla f_1^n|^2 dx & \leq \int_\Omega K f_1^n (f_1^n - 1) dx, \\
0 & \leq f_1^n(t, x) \leq 1.
\end{align*}
\]

which imply that

\[
\begin{align*}
f_1^n \text{ is bounded in } L^\infty((0, T) \times \Omega) \cap W^{1,2}((0, T) \times \Omega), \\
f_1^n & \to f_1 \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)).
\end{align*}
\]

Property (5.18) will be very useful in the sequel in obtaining bounds for \( \rho_1, \nabla \rho_1 \) and other quantities of interest.

Now standard compactness arguments yield

\( \rho_{1n} \to \rho_1 \text{ in } C([0, T]; W^{-1,2}(K)) \), for any bounded \( K \subset \Omega \)

and so

\[
\begin{align*}
\rho_{1n} u_n & \to \rho_1 u \text{ *-weakly in } L^\infty(0, T; L^{\frac{2N}{N+2}}(\Omega)), \\
\rho_{1n} \theta_n & \to \rho_1 \theta \text{ *-weakly in } L^\infty(0, T; L^{\frac{2N}{N+2}}(\Omega)).
\end{align*}
\]

Using now the fact that \( \{ \rho_n \log(\theta_n), \rho_{1n} \log(\theta_n) \} \) satisfy the entropy inequality (5.6) we deduce

\[
\begin{align*}
\rho_n \log(\theta_n) & \text{ bounded in } L^\infty(0, T; L^1(\Omega)) \cap L^2(0, T; L^{\frac{6\alpha}{3+\alpha}}(\Omega)) \\
\rho_{1n} u_n \log(\theta_n) & \text{ bounded in } L^2(0, T; L^{\frac{6\alpha}{3+\alpha}}(\Omega)).
\end{align*}
\]

and

\[
\begin{align*}
\rho_{1n} \log(\theta_n) & \text{ bounded in } L^\infty(0, T; L^1(\Omega)) \cap L^2(0, T; L^{\frac{6\alpha}{3+\alpha}}(\Omega)) \\
\rho_{1n} u_n \log(\theta_n) & \text{ bounded in } L^2(0, T; L^{\frac{6\alpha}{3+\alpha}}(\Omega)).
\end{align*}
\]

The last two estimates can be improved using (5.18). Now starting from the regularized species conservation equation, multiplying it by \( \rho_{1n} \) and integrating over \( \Omega \) we
get, in the spirit of our earlier discussion, a species density gradient estimate, which can in fact be improved using (5.18) and standard $L^p$ theory for parabolic equations to obtain:

**Lemma 5.3.** There exists $p > 1$ such that
\[
\partial_t \rho_n, \ \Delta \rho_n \quad \text{are bounded in } L^p((0,T) \times \Omega), \quad p > 1,
\]
independently of $n$. Consequently, the limit functions $\rho, u$ satisfy (4.4) a.e on $(0,T) \times \Omega$ whereas the boundary condition and initial condition hold in the sense of traces.

### 5.5. Pointwise convergence for the temperature

The following lemma will be very useful in the sequel.

**Lemma 5.4.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded Lipschitz domain. Let $\{v_n\}$ be a sequence of functions bounded in
\[
L^2(0,T; L^q(\Omega)) \cap L^\infty(0,T; L^1(\Omega)) \quad q > \frac{2N}{N + 2}.
\]
Furthermore assume that
\[
\partial_t v_n \geq g_n \quad \text{in } \mathcal{D}'((0,T) \times \Omega)
\]
where the distributions $g_n$ are bounded in the space $L^1(0,T; W^{-m,p}(\Omega))$, for $m \geq 1$, $p > 1$. Then
\[
v_n \rightarrow v \quad \text{in } L^2(0,T; W^{-1,2}(\Omega))
\]
passing into a subsequence as the case may be.

**Proof:** The proof is given in Lemma 6.3 of Chapter 6 in [22].

By a direct application of the Lemma 5.4 to the sequence
\[
\left\{ \frac{4a}{3} \theta^3 + \frac{c_v}{m_2} \rho_n \log(\theta_n) + Lc_v \rho_n \log(\theta_n) \right\},
\]
we get
\[
\begin{align*}
\left\{ \frac{4a}{3} \theta^3 + \frac{c_v}{m_2} \rho_n \log(\theta_n) + Lc_v \rho_n \log(\theta_n) \right\} \\
\left\{ \frac{4a}{3} \tilde{\theta}^3 + \frac{c_v}{m_2} \rho \log(\theta) + Lc_v \rho \log(\theta) \right\}
\end{align*}
\]
weakly in $L^2(0,T; W^{-1,2}(\Omega))$

Using now (5.16) we can conclude
\[
\begin{align*}
\int_0^T \int_\Omega \left( \frac{4a}{3} \theta^3 + \frac{c_v}{m_2} \rho_n \log(\theta_n) + L\rho_n \log(\theta_n) \right) \theta_n dx dt \\
\int_0^T \int_\Omega \left( \frac{4a}{3} \tilde{\theta}^3 + \frac{c_v}{m_2} \rho \log(\theta) + L\rho \log(\theta) \right) \theta dx dt
\end{align*}
\]
Since the function
\[
y \rightarrow \left\{ \frac{4a}{3} y^3 + \frac{c_v}{m_2} \rho \log(y) + Lc_v \rho \log(y) \right\}
\]
is nondecreasing we have
\[ \theta_n \longrightarrow \theta \quad \text{strongly in } L^1((0, T) \times \Omega). \] \hspace{1cm} (5.19)

Now by interpolation arguments we have that
\[ \begin{cases} \theta_n \longrightarrow \theta & \text{strongly in } L^p((0, T) \times \Omega) \quad \text{for } p > 4 \\ S_n \longrightarrow S & \text{weakly in } L^q((0, T) \times \Omega) \quad \text{for } q > 1. \end{cases} \] \hspace{1cm} (5.20)

Similarly we get
\[ \rho_n \longrightarrow \rho \quad \text{in } L^p((0, T) \times \Omega) \quad \text{for } p > \beta. \] \hspace{1cm} (5.21)

By using the same argument as in \cite{12} we have
\[ \rho_n u_n \longrightarrow \rho u \quad \text{in } C([0, T]; L^{\frac{2\alpha}{\alpha+1}}(\Omega)), \]

which allows us to pass into to limit and to get that the limit function \( \rho, u, \theta \) satisfy (4.2) in \( D'((0, T) \times \Omega) \).

We proceed multiplying the inequality (5.17) by a function \( \psi \in C^\infty[0, T], \psi(0) = 1, \psi(T) = 0, \partial_t \psi \leq 0 \) and integrating by parts to obtain
\[
\begin{align*}
\int_0^T \int_\Omega (\partial_t \psi) \left( \frac{1}{2} |f_1|^2 \right) dx dt + \int_0^T \int_\Omega \psi D(\theta) |\nabla f_1|^2 dx dt & \leq \\
\int_0^T \int_\Omega \psi K f_1^2 (f_1 - 1) dx dt + \int_\Omega \frac{1}{2} \rho_0 |f_1|^2 dx.
\end{align*}
\] \hspace{1cm} (5.22)

In the same way we can let \( n \to \infty \) in the energy inequality (5.5) in order to get
\[
\begin{align*}
- \int_0^T \int_\Omega \partial_t \psi \left( \frac{1}{2} \rho |u|^2 + \rho P_m(\rho) + \frac{\delta}{\beta - 1} \rho^\beta + \rho f_1 + a \theta^4 + \frac{c_v}{m_2} \rho \theta + L \rho \theta \right) dxdt & = \\
= \int_\Omega \left( \frac{1}{2} \frac{m_0}{\rho_0} \rho_0 \delta \rho P_m(\rho_0, \delta) + \frac{\delta}{\beta - 1} \rho_0 \delta + \rho_0 \delta f_1 \theta_{0, \delta} + a \theta_{0, \delta}^4 \right) dx + \\
& \int_\Omega \frac{c_v}{m_2} \rho_0 \delta \theta_{0, \delta} + L \rho_0 \delta \theta_{0, \delta} dx + \int_0^T \int_\Omega \psi (p_0(\rho) \text{div} u + f_1 (f_1 - 1) - \varepsilon \nabla \rho \nabla f_1) dxdt
\end{align*}
\] \hspace{1cm} (5.23)

for any \( \psi \in C^\infty[0, T], \psi(0) = 1, \psi(T) = 0, \partial_t \psi \leq 0 \). The following two lemmas will be useful in the sequel.

**Lemma 5.5.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded Lipschitz domain. Suppose that \( \rho \) is a given nonnegative function satisfying
\[
0 < M \leq \int_\Omega \rho dx, \quad \int_\Omega \rho^\beta dx < K, \quad \beta > \frac{2N}{N+2}.
\]

(a) Then the following two statements are equivalent:

i) The function \( \theta \) is strictly positive a.e. on \( \Omega \),
\[
\rho |\log(\theta)| \in L^1(\Omega), \quad \frac{\nabla \theta}{\theta} \in L^2(\Omega).
\]
ii) The function $\log(\theta)$ belongs to the Sobolev space $W^{1,2}(\Omega)$. Moreover, if this is the case, then

$$\nabla \log(\theta) = \frac{\nabla \theta}{\theta}, \text{ a.e. on } \Omega.$$ 

(b) The following two statements are equivalent:

i) Let $f_1, \rho_1$ nonnegative functions such that $\rho_1 = \rho f_1$. The quantity

$$f_1|\log(\rho_1 \theta)| \in L^1(\Omega), \quad \frac{\nabla(\rho_1 \theta)}{\rho_1 \theta} \in L^2(\Omega)$$

and the function $\rho_1 \theta$ is strictly positive a.e. on $\Omega$ provided $f_1$ is positive.

ii) The function $\log(\rho_1 \theta)$ belongs to the Sobolev space $W^{1,2}(\Omega)$. Moreover,

$$\nabla \log(\rho_1 \theta) = \frac{\nabla \rho_1}{\rho_1} + \frac{\nabla \theta}{\theta} \text{ a.e. on } \Omega.$$

Proof: The proof of part (a) follows using similar line of arguments as the ones given in [14]. We shall only give some comments on part (b). Let $\log (\rho_1 \theta) \in W^{1,2}(\Omega)$ then it follows that the product $\rho_1 \theta$ is positive a.e. on $\Omega$. Then one can use the Sobolev embedding theorem to conclude that

$$f_1|\log(\rho_1 \theta)| \in L^1(\Omega) \text{ is integrable.}$$

The converse follows easily (see also [14]).

Lemma 5.6. Let $\theta_n \to \theta$ in $L^2((0,T) \times \Omega)$, and $\log(\theta_n) \to \log(\theta)$ weakly in $L^2((0,T)) \times \Omega$. Then $\theta$ is strictly positive a.e. on $(0,T) \times \Omega$, and $\log(\theta) = \log(\theta)$. Proof: For the proof we refer the reader to [14].

Using the Lemmas 5.5 and 5.6, the estimates in (5.11)-(5.15), relations (1.26) and Lemmas 5.1, 5.3, we can pass into the limit in the entropy inequality (5.6) to get

$$\int_0^T \int_\Omega \left\{ \partial_t \varphi[\rho(s_F + s_{f_1})] + \nabla \varphi[\rho(s_F + s_{f_1})] - \left( \frac{\kappa C(\theta) + \sigma \theta^3}{\theta} \nabla \varphi \right) \nabla \varphi \right\} dx dt$$

$$\leq \int_\Omega \varepsilon \left( \frac{c_u}{m_2} \nabla \varphi[\log(\theta) - 1] \nabla \rho + \nabla f_{1n} \nabla \rho_{1n} \right) dx$$

$$- \int_0^T \int_\Omega \varphi \left( \frac{c_v}{m_2} \nabla \frac{\kappa C(\theta) + \sigma \theta^3}{\theta^3} \nabla \theta \right) dx dt$$

$$- \int_0^T \int_\Omega \varphi \left( \frac{c_v}{m_2} \log(\theta) - 1 \right) K f_1(f_1 - 1) dx dt$$

$$- \int_\Omega \varphi(0) \rho_{0,s}(s_{F_{0,s}} + s_{f_{0,s}}) dx.$$ (5.24)

for any test function $\varphi, \varphi \in C^\infty([0,T] \times \Omega), \varphi \geq 0, \varphi(T) = 0$.

6. The vanishing viscosity limit. Our aim in this section is to let the artificial viscosity $\varepsilon$ go to zero in the family of approximate solutions $\{\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, f_{1\varepsilon}\}$ constructed in the previous section. We remark that during this process we expect loss of regularity of $\rho_\varepsilon$ due to the fact that the parabolic regularization $\varepsilon \Delta \rho_\varepsilon$ now vanishes. The main difficulty is to establish the strong convergence of the density $\rho_\varepsilon$. 
6.1. **Refined pressure estimates.** Our goal in this section is to improve the estimates on the pressure, which so far yield only that $p$ is bounded in the non reflexive space $L^\infty(0, T; L^1(\Omega))$, by improving the integrability properties of the modified pressure

$$ p(\rho_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon, f_\varepsilon) + \delta \rho_\varepsilon^\beta. $$

We employ the method known as *multipliers technique* introduced by Lions [32] in the barotropic case and Feireisl in [22] for handling the full system. This method involves placing a particular choice of test functions, namely

$$ \varphi(t, x) = \chi(t)B[\rho_\varepsilon^\nu] \quad \chi \in D(0, T), \ 0 \leq \chi \leq 1 $$

in the weak formulation of the momentum equation (4.2).

The quantity $B[v]$ represents a suitable set of solutions to the problem (see [22])

$$ \text{div } (B[v]) = v - \frac{1}{|\Omega|} \int_\Omega v dx \quad \text{in } \Omega, \ B[v]|_{\partial \Omega} = 0. $$

Integrating the modified pressure against suitable test functions yields the following integral relation

$$ \int_0^T \int_\Omega \left( p_\varepsilon(\rho_\varepsilon) + \frac{a}{3} \theta_\varepsilon^4 + \frac{R}{m_2} \theta_\varepsilon \rho_\varepsilon + L \rho_\varepsilon, \theta_\varepsilon + \delta \rho_\varepsilon^\beta \right) \rho_\varepsilon^\nu dx \, dt = \sum_{j=1}^8 \mathcal{L}_{j\varepsilon}, \quad (6.1) $$

where $\nu$ is a positive constant and

$$ \begin{align*}
\mathcal{L}_{1\varepsilon} &= \int_0^T \int_\Omega \left( \chi \left( \int_\Omega p_\varepsilon(\rho_\varepsilon) + \frac{a}{3} \theta_\varepsilon^4 + \frac{R}{m_2} \theta_\varepsilon + L \rho_\varepsilon, \theta_\varepsilon + \delta \rho_\varepsilon^\beta \right) dt \\
\mathcal{L}_{2\varepsilon} &= \int_0^T \int_\Omega \nabla B \left[ \rho_\varepsilon - \frac{1}{|\Omega|} \right] dx \, dt, \\
\mathcal{L}_{3\varepsilon} &= -\int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla B \left[ \rho_\varepsilon - \frac{1}{|\Omega|} \right] dx \, dt, \\
\mathcal{L}_{4\varepsilon} &= \varepsilon \int_0^T \int_\Omega \nabla (\varepsilon \mathbf{u}_\varepsilon \nabla \rho_\varepsilon) \cdot B \left[ \rho_\varepsilon - \frac{1}{|\Omega|} \right] dx \, dt, \\
\mathcal{L}_{5\varepsilon} &= -\int_0^T \int_\Omega \rho_\varepsilon \nabla \psi_\varepsilon \cdot B \left[ \rho_\varepsilon - \frac{1}{|\Omega|} \right] dx \, dt, \\
\mathcal{L}_{6\varepsilon} &= \int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot B \left[ \rho_\varepsilon - \frac{1}{|\Omega|} \right] dx \, dt, \\
\mathcal{L}_{7\varepsilon} &= -\varepsilon \int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot B(\Delta \rho_\varepsilon) dx \, dt \\
\mathcal{L}_{8\varepsilon} &= \int_0^T \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot B(\text{div}(\rho_\varepsilon \mathbf{u}_\varepsilon)) dx \, dt.
\end{align*} $$

Taking now into consideration that the estimates in (5.11)-(5.15) hold true and that $0 \leq f_{\varepsilon} \leq 1$ and following the same line of arguments as in [14], [22] we show that the quantities $S_\varepsilon, \rho_\varepsilon, \rho_\varepsilon \mathbf{u} \otimes \mathbf{u}$ are bounded in $L^p((0, T) \times \Omega)$ for a certain $p > 1$. Using now the fact that operator

$$ B : \left\{ f \in L^p(\Omega) \mid \int_\Omega f dx = 0 \right\} \to [W^1_0(\Omega)]^3 $$

introduced earlier is a bounded linear operator, namely

$$ \|B[f]\|_{[W^1_0(\Omega)]^3} \leq c(p)\|f\|_{L^p(\Omega)}, \quad \text{for any } 1 < p < \infty, $$

we conclude that the integrals $\mathcal{L}_{1\varepsilon} - \mathcal{L}_{4\varepsilon}$ and $\mathcal{L}_{5\varepsilon}$ are bounded uniformly with $\varepsilon$. The integral $\mathcal{L}_{4\varepsilon}$ can be controlled using a standard density gradient estimate (see
while the other integrals can be controlled using standard embedding theorems and further properties of the operator $B$. Therefore, there exists $\nu > 0$ and a positive constant $c(\delta)$ independent of $\epsilon$ by which

$$
\int^T_0 \int_\Omega \left( p_\epsilon(\rho_\epsilon) + \frac{a}{3} \theta_\epsilon^2 + \frac{R}{m_2} \theta_\epsilon \rho_\epsilon + L \rho_1 \theta_\epsilon + \delta \rho_\epsilon^2 \right) \rho_\epsilon \, dx \, dt \leq c(\delta).
$$

Moreover,

$$
\int^T_0 \int_\Omega \rho_\delta^{\gamma+\nu} + \delta \rho_\delta^{\beta+\nu} \, dx \, dt \leq c(\delta).
$$

6.2. **Strong convergence of the temperature.** Taking into consideration estimates (5.11)-(5.15) we may now assume that

$$
\begin{align*}
\theta_\epsilon & \rightharpoonup \theta \quad \text{weakly in} \quad L^2(0, T; W^{1,2}(\Omega)), \\
\theta_\epsilon & \rightharpoonup \theta \quad \text{weakly-* in} \quad L^\infty(0, T; L^4(\Omega)), \\
\log(\theta_\epsilon) & \rightharpoonup \log(\theta) \quad \text{weakly in} \quad L^2(0, T; W^{1,2}(\Omega)),
\end{align*}
$$

$$
\begin{align*}
\rho_\epsilon & \rightarrow \rho \quad \text{in} \quad C([0, T], L^2_{\text{weak}}(\Omega)). \\
u_\epsilon & \rightarrow \mathbf{u} \quad \text{weakly in} \quad L^b(0, T; W^{1,b}_0(\Omega)), \\
\rho_\epsilon \mathbf{u}_\epsilon & \rightarrow \rho \mathbf{u} \quad \text{in} \quad C([0, T], L^{2\beta}_{\text{weak}}(\Omega)), \\
f_\epsilon & \rightarrow f_1 \quad \text{weakly in} \quad L^2(0, T; W^{1,2}(\Omega)).
\end{align*}
$$

(6.2)

Combining (6.2) and (6.3) we obtain

$$
\begin{align*}
\rho_\epsilon \log(\theta_\epsilon) \mathbf{u}_\epsilon & \rightharpoonup \rho \log(\theta) \mathbf{u} \quad \text{weakly in} \quad L^p((0, T) \times \Omega) \quad \text{for} \quad p > 1, \\
\rho_1 \log(\theta_\epsilon) \mathbf{u}_\epsilon & \rightharpoonup \rho_1 \log(\theta) \mathbf{u} \quad \text{weakly in} \quad L^p((0, T) \times \Omega) \quad \text{for} \quad p > 1.
\end{align*}
$$

(6.4)

Following a similar procedure to the one of the previous section we end up with

$$
\theta_\epsilon \rightharpoonup \theta \quad \text{strong in} \quad L^2((0, T) \times \Omega).
$$

(6.5)

6.3. **Strong convergence for the density.** The aim in this section is to show the strong convergence of density sequence $\rho_\epsilon$. This involves understanding the time evolution of the defect measure

$$
\mathcal{B}_{dft}[\rho_\epsilon - \rho] = \int_\Omega [\rho \log(\rho)(t) - \rho \log(\rho(t))] \, dx.
$$

(6.6)

Consider now the renormalized version of the regularized continuity equation (4.1),

$$
\partial_t \beta(\rho_\epsilon) + \text{div}(\beta(\rho_\epsilon) \mathbf{u}_\epsilon) + (\beta'(\rho_\epsilon) \rho_\epsilon - \beta(\rho_\epsilon)) \text{div} \mathbf{u}_\epsilon = \\
\varepsilon \text{div}(\chi_\Omega \nabla \beta(\rho_\epsilon)) - \varepsilon \chi_\Omega \beta''(\rho_\epsilon) |\nabla \rho_\epsilon|^2 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^3),
$$

with $\beta \in C^2[0, \infty)$, $\beta(0) = 0$, a convex function, $\beta', \beta''$ bounded, and $\chi_\Omega$ the characteristic function on $\Omega$. By a suitable approximation of $z \rightarrow z \log z$ by smooth convex functions we deduce for $\epsilon \rightarrow 0$ (see also [14] [13])

$$
\mathcal{B}_{dft}[\rho_\epsilon - \rho] \leq \int^T_0 \int_\Omega [\rho \text{div} \mathbf{u} - \rho \text{div} \mathbf{u}] \, dx \, dt
$$

(6.7)
for a.e. \( \tau \in [0, T] \).

In the sequel we employ the **multipliers technique** as in Feireisl [22] and Lions [32], which involves placing the quantities

\[
\varphi(t, x) = \psi(t)\eta(x)(\nabla \Delta^{-1})[\rho_\varepsilon], \quad \psi \in \mathcal{D}(0, T), \quad \eta \in \mathcal{D}(\Omega)
\]

as test functions in the approximate momentum equation (4.2). Using now the smoothing properties of the operator \( \{\nabla \Delta^{-1}\} \) we get (see also [14], [13])

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \psi \eta \left[ \rho_\varepsilon(\rho_\varepsilon) + \frac{R}{m_2} \theta_\varepsilon \rho_\varepsilon + \frac{L \rho_1}{m_2} \theta_\varepsilon + \frac{\delta}{2} \rho_\varepsilon^2 \right. \\
\left. - \left( \left( \frac{\zeta(\theta_\varepsilon)}{3} \right) - \frac{2}{3} \mu(\theta_\varepsilon) + 2 \mu(\theta_\varepsilon) \right) \text{div} \mathbf{u}_\varepsilon \right] \rho \, dt
\]

\[
= \int_0^T \int_\Omega \psi \eta \left[ \rho_\varepsilon(\rho) + \frac{R}{m_2} \theta \rho(\rho) + \frac{L \rho_1}{m_2} \theta + \frac{\delta}{2} \rho^2 \right. \\
\left. - \left( \left( \frac{\zeta(\theta)}{3} \right) - \frac{2}{3} \mu(\theta) + 2 \mu(\theta) \right) \text{div} \mathbf{u} \right] \rho \, dt
\]

\[
+ (I^1 - \lim_{\varepsilon \to 0} I^1_\varepsilon) + 2(\lim_{\varepsilon \to 0} I^2_\varepsilon - I^2), \tag{6.8}
\]

with

\[
I^1 = \int_0^T \int_\Omega \psi \eta \mathbf{u} \cdot (\rho \mathcal{R}[\rho \mathbf{u}] - \mathcal{R}[\rho](\rho \mathbf{u})) \, dx \, dt,
\]

\[
I^1_\varepsilon = \int_0^T \int_\Omega \psi \eta \mathbf{u}_\varepsilon \cdot (\rho \mathcal{R}[\rho \mathbf{u}_\varepsilon] - \mathcal{R}[\rho_\varepsilon](\rho_\varepsilon \mathbf{u}_\varepsilon)) \, dx \, dt,
\]

\[
I^2 = \int_0^T \int_\Omega \psi \mathcal{R}[\eta \mu(\theta) \mathbf{u}] - \eta \mu(\theta) \mathcal{R}[\mathbf{u}] \rho \, dx \, dt,
\]

\[
I^2_\varepsilon = \int_0^T \int_\Omega \psi \mathcal{R}[\eta \mu(\theta_\varepsilon) \mathbf{u}_\varepsilon] - \eta \mu(\theta_\varepsilon) \mathcal{R}[\mathbf{u}_\varepsilon] \rho \, dx \, dt.
\]

where

\[
\mathcal{R}[A] = \sum_{i,j} \mathcal{R}_{i,j}[A_{i,j}], \quad \mathcal{R} = \mathcal{R}_{i,j}[v] = \mathcal{F}^{-1}_{x-\xi} \left[ \frac{\xi_i \xi_j}{|\xi|^2} \mathcal{F} \mathbf{u} \right].
\]

Using now the continuity property of the bilinear form

\[
[v, w] \to v \mathcal{R}[w] - \mathcal{R}[v]w
\]

one obtains as in [24], [22], [32] that

\[
\lim_{\varepsilon \to 0} I^1_\varepsilon = I^1.
\]

The convergence

\[
\lim_{\varepsilon \to 0} I^2_\varepsilon = I^2
\]

is obtained following the analysis presented in Feireisl [22], [15] in the spirit of Coffman and Meyer [8].

Now relation (6.8) together with the strong convergence of \( \{\theta_\varepsilon\} \) yields

\[
\rho \text{div} \mathbf{u} - \overline{\rho \text{div} \mathbf{u}} \leq \frac{1}{\zeta(\theta) - \frac{2}{3} + \mu(\theta)} (P_1 + P_2 + P_3),
\]
where
\[ \mathcal{P}_1 = p_e(\rho) - p_i(\rho), \quad \mathcal{P}_2 = \frac{R \theta}{m_2} (p_\theta(\rho) - p_\theta(\bar{\rho})) + L \theta (\rho_1 - \bar{\rho}_1), \quad \mathcal{P}_3 = \rho^2 \rho - \rho^{2+1}. \]

We can see immediately that \( \mathcal{P}_2 \leq 0, \mathcal{P}_3 \leq 0, \) while
\[ \mathcal{P}_1 \leq p_b(\rho) - p_b(\bar{\rho}) \]
where \( p_b \) is the bounded, nonmonotone component of the elastic pressure. Using (6.7) we get
\[ \int_\Omega \left( \rho \log (\rho) - \rho \log (\bar{\rho}) \right) (s) dx \leq \frac{1}{\mu} \int_0^s \int_\Omega \frac{p_b(\rho) - p_b(\bar{\rho})}{\rho} dx dt, \]
which yields
\[ \mathcal{B}_{df} [\rho_e - \rho] \leq \frac{\Lambda}{\mu} \int_0^T \mathcal{B}_{df} [\rho_e - \rho] dx, \]
and consequently
\[ \mathcal{B}_{df} [\rho_e - \rho] = 0, \]
which implies that
\[ \rho_e \rightarrow \rho \quad \text{in} \quad L^1((0,T) \times \Omega). \quad (6.9) \]

6.4. The limit process in the field equations \((\varepsilon \rightarrow 0)\). We are now ready to let \( \varepsilon \rightarrow 0 \) in the field equations. By the aid of the estimates obtained above, we have passing to a subsequence if needed, that
\[ \begin{cases} 
\rho_e \rightarrow \rho & \text{in} \quad C([0,T]; L^1(\Omega)), \\
\mathbf{u}_e \rightarrow \mathbf{u} & \text{weakly in} \quad L^2(0,T; W^{1,2}_0(\Omega; \mathbb{R}^3)), \\
\varepsilon \text{div}(\chi_\Omega \nabla \rho_e) \rightarrow 0 & \text{in} \quad L^2(0,T; W^{-1,2}(\mathbb{R}^N)),
\end{cases} \quad (6.10) \]
therefore, the limit functions \( \rho, \mathbf{u} \) satisfy the continuity equation in the sense of distributions. From the energy estimates established above we have,
\[ \varepsilon \nabla \mathbf{u}_e \nabla \rho_e \rightarrow 0, \quad \varepsilon \nabla \rho_e \nabla f_{1e} \rightarrow 0, \quad \varepsilon \nabla \mathbf{u}_e \nabla \rho_{1e} \rightarrow 0 \quad \text{in} \quad L^1(0,T; L^1(\Omega)). \]

Having in mind the estimates (6.10) we get
\[ \rho_e \mathbf{u}_e \rightarrow \rho \mathbf{u}, \quad \rho_{1e} \rightarrow \rho_1 \quad \text{in} \quad C([0,T]; L^1_{weak}(\Omega)). \]

The limit function \( \rho, \mathbf{u}, \theta \) and \( f_1 \) satisfy in the sense of distributions \( \mathcal{D}'((0,T) \times \Omega) \) the momentum equation
\[ \partial_t (\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla \left( \frac{p_e(\rho)}{m_2} \theta p_\theta(\rho) + \frac{a}{3} \theta^4 + LR \rho_1 \theta + \delta \rho^{2+1} \right) = \text{div} \mathbf{S}. \]

The species conservation equation is also verified in \( \mathcal{D}'((0,T) \times \Omega) \) by the limit function sequence \( \rho, \mathbf{u}, \theta \) and \( f_1 \) by virtue of (5.18) and the estimates above. Now passing
into the limit in the energy equality (5.23) comes as a consequence of the estimates established earlier and so we recover the total energy balance,

\[ -\int_0^T \int_\Omega \partial_t \psi \left( \frac{1}{2} |u|^2 + \rho P_e(\rho) + \frac{\delta}{\beta - 1} \rho^\beta + a \theta^4 + \frac{R}{m_2} \rho \theta + LR \rho \theta \right) \, dx \, dt \]

\[ = \int_\Omega \left\{ \frac{1}{2} \rho_{0,\delta} + \rho_{0,\delta} \rho_{0,\delta} + \frac{\delta}{\beta - 1} \rho_{0,\delta}^\beta + a \theta^4_{0,\delta} + \frac{R}{m_2} \rho_{0,\delta} \theta_{0,\delta} + LR \rho_{0,\delta} \theta_{0,\delta} \right\} \, dx \]

(6.11)

for any \( \psi \in C^\infty[0, T], \psi(0) = 1, \psi(T) = 0, \partial_t \psi \leq 0 \). Similarly sending \( \varepsilon \to 0 \) in (5.24)

\[ \int_0^T \int_\Omega \partial_t \psi \left( \rho(s_F + s_{f_1}) + \nabla \psi \left( \rho(s_F + s_{f_1}) \right) - \left( \frac{\kappa_C(\theta) + \sigma \theta^3}{\theta} \right) \nabla \psi \right) \, dx \, dt \]

\[ - \int_0^T \int_\Omega \varphi \left( \frac{\kappa_C(\theta)}{\theta} + \frac{\sigma \theta^3}{\theta^3} \right) \nabla \psi \, dx \, dt \]

\[ - \int_0^T \int_\Omega \varphi \left( \frac{c_0 m_2}{m_2} (\log(\theta) - 1) K f_1 (f_1 - 1) \right) \, dx \, dt \]

\[ - \int_\Omega \varphi(0) \rho_{0,\delta}(s_{F,\delta} + s_{f_1,\delta}) \, dx, \]

(6.12)

for any test function \( \varphi, \varphi \in C^\infty([0, T] \times \Omega), \varphi \geq 0, \varphi(T) = 0 \).

7. Passing to the limit in the artificial pressure term. In this last part we pass into the limit for \( \delta \to 0 \) in the sequence \( \rho_\delta, u_\delta, \theta_\delta, f_1, \) of the approximate solutions constructed in the previous section. Again in this part the central issue is to recover strong compactness for \( \rho_\delta \) and \( \theta_\delta \).

Taking into consideration the energy equality (6.11) we have

\[
\begin{align*}
\rho_\delta & \in L^\infty(0, T; L^\gamma(\Omega)), \\
\sqrt{\lambda_\delta} u_\delta & \in L^\infty(0, T; L^2(\Omega)), \\
\sqrt{\lambda_\delta} f_1 & \in L^\infty(0, T; L^2(\Omega))
\end{align*}
\]

(7.1)

Moreover as in Section 3 we get

\[
\begin{align*}
\rho_\delta^{3/2} & \in L^2(0, T; W^{1,2}(\Omega)), \\
\log(\theta_\delta) & \in L^2(0, T; W^{1,2}(\Omega))
\end{align*}
\]

(7.2)

By applying now the same procedure as in Section 5 we get the following refined estimate for \( \rho_\delta \)

\[
\rho_\delta^{\gamma + \nu} + \delta \rho_\delta^{\gamma + \nu} \text{ is bounded in } L^1((0, T) \times \Omega), \nu > 1.
\]

(7.3)

Now by virtue of (7.1)-(7.2) we can suppose

\[
\begin{align*}
\rho_\delta & \to \rho \quad \text{in } C([0, T], L^\infty(\Omega)), \\
u_\delta & \to \nu \quad \text{weakly in } L^b(0, T; W^{1,2}_0(\Omega)),
\end{align*}
\]

(7.4)

where \( \rho, \nu \) satisfy the equation (1.28) in \( D^t((0, T) \times \mathbb{R}^3) \). We have also

\[
\begin{align*}
\rho_\delta u_\delta & \to \rho u \quad \text{in } C([0, T], L^\infty(\Omega)), \\
\log(\theta_\delta) & \to \log(\theta) \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \\
\rho_\delta \log(\theta_\delta) & \to \rho \log(\theta) \quad \text{weakly in } L^2(0, T; L^{s_0}(\Omega)), \\
\rho_\delta \log(\theta_\delta) u_\delta & \to \rho \log(\theta) u \quad \text{weakly in } L^2(0, T; L^{s_0}(\Omega)),
\end{align*}
\]

(7.5)
whereas estimates involving \( \rho_1 \) are obtained using (5.18), (7.4) and (7.5).

7.1. Pointwise convergence for the temperature. Next we show the strong convergence of the temperature sequence. In analogy with the above analysis, the entropy inequality (6.12) in combination with Lemma 5.4 yield

\[
\begin{align*}
\frac{4}{3} \theta_3^3 + \frac{c_v}{m_2} \rho_3 \log(\theta_3) - \frac{4}{3} \bar{P} \rho_3(\rho_3) + \text{LC}_{\rho_1} \rho_3 \log(\theta_3) - LR \rho_1 \log(\rho_1) \\
\frac{4}{3} \bar{\theta}_3^3 + \frac{c_v}{m_2} (\rho \log(\theta) - \rho \bar{P} \rho(\rho)) + \text{LC}_{\rho_1} \rho \log(\theta) - LR \rho_1 \log(\rho_1)
\end{align*}
\]

in \( L^2(0, T; W^{-1,2}(\Omega)) \).

We remark that in the present context

\[
\rho_3 \bar{P} \rho(\rho_3) \text{ is bounded in } L^\infty(0, T; L^\infty(\Omega)), \text{ with } \frac{\gamma}{G} > \frac{4}{3}.
\]

In particular we have

\[
\int_0^T \int_\Omega \left( \frac{4}{3} \theta_3^3 + \frac{c_v}{m_2} \rho_3 \log(\theta_3) - \frac{4}{3} \bar{P} \rho_3(\rho_3) + \text{LC}_{\rho_1} \rho_3 \log(\theta_3) - LR \rho_1 \log(\rho_1) \right) \theta_3 \, dx \, dt
\]

\[
\int_0^T \int_\Omega \left( \frac{4}{3} \bar{\theta}_3^3 + \frac{c_v}{m_2} (\rho \log(\theta) - \rho \bar{P} \rho(\rho)) + \text{LC}_{\rho_1} \rho \log(\theta) - LR \rho_1 \log(\rho_1) \right) \theta_3 \, dx \, dt,
\]

where, we have used (6.3) and (6.4) to obtain that as \( \delta \to 0 \)

\[
\lim_{\delta \to 0} \int_0^T \int_\Omega \rho_3 \log(\theta_3) \theta_3 \, dx \, dt = \int_0^T \int_\Omega \rho \log(\theta) \rho \, dx \, dt,
\]

\[
\lim_{\delta \to 0} \int_0^T \int_\Omega \rho_1 \log(\theta_3) \theta_3 \, dx \, dt = \int_0^T \int_\Omega \rho_1 \log(\theta) \rho_1 \, dx \, dt.
\]

In addition, taking into consideration that \( \rho \) is a renormalized solution of the continuity equation and using a standard approximation argument we get

\[
\begin{align*}
\rho_3 \bar{P} \rho(\rho_3) &\rightarrow \rho \bar{P} \rho(\rho) \text{ in } C([0, T]; L^\infty(\Omega)), \\
\rho_1 \log(\rho_1) &\rightarrow \rho_1 \log(\rho_1) \text{ in } C([0, T]; L^{\infty}(\Omega)) \text{ for } q \geq 1.
\end{align*}
\]

Therefore (6.3), (6.4) imply

\[
\int_0^T \int_\Omega \left( \frac{4}{3} \theta_3^3 + \frac{c_v}{m_2} \rho_3 \log(\theta_3) + \text{LC}_{\rho_1} \rho_3 \log(\theta_3) \right) \theta_3 \, dx \, dt
\]

\[
\int_0^T \int_\Omega \left( \frac{4}{3} \bar{\theta}_3^3 + \frac{c_v}{m_2} (\rho \log(\theta) + \text{LC}_{\rho_1} \rho \log(\theta)) \theta \, dx \, dt,
\]

which implies

\[
\theta_3 \longrightarrow \theta \text{ in } L^2((0, T) \times \Omega).
\]

7.2. Pointwise convergence for the density. In order to pass into the limit we need the strong convergence of the density. The main part consists in showing that the oscillation defect measure \( \text{osc}_{\beta+1}[\rho_3 \rightarrow \rho] \) defined by

\[
\text{osc}_{\beta+1}[\rho_3 \rightarrow \rho](0, T) = \sup_{k \geq 1} \left( \lim_{\delta \to 0} \sup_{k \geq 1} \int_0^T \int_\Omega |T_k(\rho_3) - T_k(\rho)|^{\beta+1} \, dx \, dt \right)
\]

(7.7)
is bounded. Here $T_k(\rho)$ are cut-off functions

$$T_k(y) = T\left(\frac{y}{k}\right) \quad \text{with } T \in C^\infty(\mathbb{R}) \text{ - a concave function}$$

$$T(x) = x, \quad \text{for } 0 \leq x \leq 1, \quad T(x) = 2 \quad \text{if } y \geq 3.$$  

In order to show that $\rho, u$ represent a renormalized solution of (1.28) we have to show that oscillation defect measure associated with $\{\rho_\delta\}$ is bounded.

Taking into account that the density fraction $f_1$ is bounded we estimate the amplitude of oscillations as in [14] (see also [22]), namely we write

$$p_c(\rho) = p_c^{(c)}(\rho) + p_c^{(m)}(\rho) + p_c^{(b)}(\rho),$$

with $p_c^{(b)}$ uniformly bounded on $[0, \infty)$, $p_c^{(m)}$ nondecreasing, and $p_c^{(b)}$ a convex function satisfying

$$p_c^{(c)}(\rho) \geq a\rho^\gamma, \quad \text{with } a > 0.$$  

Next, taking into account the monotonicity property of some of the quantities present in the pressure we get

$$\frac{p_0(\rho) T_k(\rho)}{p_0(\rho) T_k(\rho)} \geq \frac{p_c^{(m)}(\rho) T_k(\rho)}{p_c^{(m)}(\rho) T_k(\rho)} \quad \text{and}$$

$$\rho_1 T_k(\rho) \geq \rho_1 T_k(\rho),$$

which yield first that

$$\limsup_{\delta \to 0} \int_0^T \int_\Omega (T_k(\rho_\delta) - T_k(\rho))^{\beta + 1} dxdt$$

$$\leq \limsup_{\delta \to 0} \int_0^T \int_\Omega p_c^{(b)}(\rho_\delta) + \left((\zeta(\theta) - \frac{2}{3}) + 2\mu(\theta)\right) \text{div}(u) |T_k(\rho_\delta) - T_k(\rho)| dxdt,$$

and in the sequel, taking into consideration the properties (2.8) of the transport coefficients and the estimates derived above and following similar line of argument as presented in [14], [22], [13], we have that

$$\text{osc}_{\beta+1}[\rho_\delta \to \rho]|(0, T) \times \Omega| < \infty.$$  

Now we use the fact that $\rho, u$ represent a renormalized solution of (1.28) on $((0, T) \times \Omega)$ (cf. Proposition 6.3 Feireisl [22]).

**Proposition 7.1.** Let $\Omega \subset \mathbb{R}^N$ be a domain. Assume that $\rho_\delta \geq 0$, $u_\delta$ is a sequence of renormalized solutions to (1.28) on $(0, T) \times \Omega$ such that

$$\rho_\delta \to \rho \quad \text{weakly-* in } L^\infty(0, T; L^\gamma(\Omega)), \quad \gamma > 1,$$

$$u_\delta \to u \quad \text{weakly in } L^r(0, T; W^{1,r}(\Omega; \mathbb{R}^N)), \quad r > 1,$$

(7.8)

where

$$\gamma > \frac{Nr}{(N+1)r-N} \quad \text{if } r < N.A$$
Furthermore assume that
\[ \text{osc}_p[\rho_\delta \to \rho][(0, T) \times \Omega] < \infty, \]
for a certain \( p \) such that
\[ \frac{1}{p} + \frac{1}{r} < 1. \]

Then, the limit functions \( \rho, u \) represent a renormalized solution of (1.28) on \((0, T) \times \Omega\). Note that in our case \( r = \frac{8}{3} > \frac{3}{2} \) and so all the requirements of Proposition 7.1 are valid. Therefore we get that
\[ \rho_\delta \to \rho \quad \text{strongly in } L^1((0, T) \times \Omega). \quad (7.9) \]

Property (7.7) implies now that the continuity equation (1.28) holds true in the sense of distribution. Furthermore, the bound (7.3) yields that
\[ \delta \rho^3 \to 0 \quad \text{in } L^{\frac{2r}{r-2}}((0, T) \times \Omega), \]
and the momentum equation (1.29) is recovered as \( \delta \to 0 \).

The species conservation equation now, can also be verified using (5.18) and the estimates in (7.5). In addition, the strong convergence of the density sequence \( \rho_\delta \) together with the estimates established above, allow us to pass into the limit both in the energy equality (6.11) and in the entropy inequality (6.12). The proof of Theorem 3.1 has now been established.

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