Abstract. The rate of convergence for vanishing viscosity approximations to hyperbolic balance laws is established. The result applies to systems that are strictly hyperbolic and genuinely nonlinear with a source term satisfying a special mechanism that induces dissipation. The proof relies on error estimates that measure the interaction of waves. Shock waves are treated by monitoring the evolution of suitable Lyapunov functionals, whereas interactions involving rarefaction waves are accommodated by employing a sharp decay estimate.

Key words. Hyperbolic balance laws, source term, vanishing viscosity approximations, genuine nonlinearity, BV estimates, semigroup, dissipative mechanisms, rate of convergence.

AMS subject classifications. 35L65, 76N10, 35B45.

1. Introduction. We consider the Cauchy problem

\[ \partial_t u + \partial_x f(u) + g(u) = 0, \]

\[ u(0, x) = \bar{u}(x), \quad x \in \mathbb{R}, \]

for a strictly hyperbolic and genuinely nonlinear system of balance laws in one space dimension. Here \( x \in \mathbb{R}, \ t \geq 0 \) and the state \( u(t, x) \) takes values in \( \mathbb{R}^n \). Also, the flux \( f \) and the source \( g \) are given smooth functions from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). System (1.1) is strictly hyperbolic, that is, the Jacobian matrix \( A(u) = Df(u) \) has \( n \) real and distinct eigenvalues

\[ \lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u), \]

known as characteristic speeds and thereby, \( n \) linearly independent right eigenvectors \( r_1(u), r_2(u), \ldots, r_n(u) \). Moreover, all characteristic fields are genuinely nonlinear, namely for each \( i = 1, \ldots, n \), the product of the gradient of the characteristic speed with the corresponding right eigenvector is nonzero. Without loss of generality, this condition can be expressed as

\[ D\lambda_i(u)r_i(u) \equiv 1, \quad i = 1, \ldots, n. \]

Solutions to system of conservation laws, i.e. \( g \equiv 0 \), have been constructed globally in time in the space of bounded variation \( BV \) by various methods: the random choice method of Glimm [18], the front tracking algorithm [5] and the vanishing viscosity method [4] under the assumption of small total variation of the initial data. An exposition of the current state of the theory of conservation laws can be found in the manuscripts [5, 15, 23, 26].
For systems of balance laws with general source term \( g \), blow-up of solutions in finite time is expected even when the initial data is of small variation. In fact, the presence of the production term \( g(u) \) results to the amplification in time of even small oscillations in the solution. Because of this feature of hyperbolic balance laws, one does not expect in general long term stability in BV. Local in time existence of BV solutions was first established by Dafermos and Hsiao [17], using the random choice method of Glimm [18] in conjunction with the operator splitting. Global existence is expected if \( g \) satisfies some special mechanism that induces dissipation. In [17], Dafermos and Hsiao constructed global BV solutions under a suitable dissipativeness assumption on \( g \) using Glimm’s scheme. Under the same dissipativeness assumption, existence as well as stability results have been established via the front tracking method by Amadori and Guerra [2]. These results [17, 2] apply to hyperbolic balance laws whose characteristic fields are either genuine nonlinear or linearly degenerate. Existence and also stability results have been established for hyperbolic balance laws with general flux, even in the case of non-conservative systems, via the method of vanishing viscosity by Christoforou [12].

The dissipativeness hypothesis on the source \( g \) is the following:

**Definition 1.1 (Dissipativeness hypothesis).** Consider a constant equilibrium solution \( u^* \) to (1.1), i.e. \( g(u^*) = 0 \). If we linearize the hyperbolic system (1.1) about \( u^* \) and then decompose the solution \( u \) along the right eigenvectors of \( A(u^*) \), the resulting linear system is

\[
v_{i,t} + \lambda_i(u^*)v_{i,x} + \sum_{j=1}^{n} B_{ij}(u^*)v_j = 0,
\]

where \( B_{ij} \) are the entries of the \( n \times n \) matrix

\[
B(u) = [r_1(u), ..., r_n(u)]^{-1}Dg(u)[r_1(u), ..., r_n(u)].
\]

The natural condition that renders the above linear system stable in \( L^1 \) is that the matrix \( B(u^*) \) is strictly column diagonally dominant, i.e. there is a positive constant \( \tilde{\beta} \) such that

\[
B_{ii}(u^*) - \sum_{j \neq i} |B_{ji}(u^*)| > \tilde{\beta} > 0 \quad \text{for all } i = 1, ..., n, \quad (1.3)
\]

which is equivalent to

\[
|I - \tau B(u^*)| < 1 - \tilde{\beta}\tau,
\]

for every \( \tau \geq 0 \) sufficiently small and \( |\cdot| \) stands for the \( l^1 \) norm of a matrix.

This hypothesis depends on the choice of the right eigenvectors \( r_i(u) \). In fact, for practical use, one can find an equivalent condition in [1] which is independent of the choice of the eigenvectors.

We consider next the family of parabolic problems

\[
\partial_t u^\varepsilon + A(u^\varepsilon) u^\varepsilon_x + g(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon. \quad (1.4)
\]

with initial data (1.2), where \( \varepsilon \) is a small positive parameter. If the initial data is of small total variation, then the solutions \( \{u^\varepsilon\} \) of system (1.4) exist for all \( t \geq 0 \), have uniformly small total variation bounded independently of \( \varepsilon \) and as \( \varepsilon \to 0^+ \), the
family \{u^\varepsilon\} converges to a unique solution of the hyperbolic problem (1.1)-(1.2). (cf. Christoforou [12], [13]).

The aim of this work is to obtain the rate of convergence of the vanishing viscosity approximations \(u^\varepsilon\) to \(u\) by estimating the \(L^1\) distance between an exact BV solution \(u(t, \cdot)\) to the balance law (1.1)-(1.2) and a viscous approximation \(u^\varepsilon(t, \cdot)\) to (1.4), (1.2) at time \(t\), i.e. establish a bound on

\[\|u^\varepsilon(t) - u(t)\|_{L^1}.\] (1.5)

Such a bound would be useful in numerical computations. We study this problem under the dissipativeness hypothesis (1.3) on the source term \(g(u)\). Our analysis could be extended to general balance laws (when hypothesis (1.3) does not hold necessarily), as long as one investigates the difference (1.5) within the time interval \(t \in [0, T]\) on which the existence of solutions to (1.1) holds true. Indeed, one of the main tools that we employ in this paper is the result by Christoforou and Trivisa in [14] on the spreading of rarefaction waves, which is also available for general source \(g\) up to \(t = T\).

It should be noted that the result in [14] yields a sharp decay estimate that relates systems of balance laws with a scalar Burgers’s equation with a forcing term. Therefore, this result allows us to establish quantitative estimates and more precisely, the convergence rate of the viscous approximations.

In the special case where (1.1) is a system of conservation laws, i.e. \(g \equiv 0\), there are available results in that direction. More precisely, Kuznetsov [21] studied the one dimensional scalar conservation law and showed that the rate of convergence is of the order

\[\|u^\varepsilon(\tau) - u(\tau, \cdot)\|_{L^1} = O(1) \cdot \varepsilon^{1/2}.\]

See also Tang and Teng [27]. In [20], Goodman and Xin consider the case of viscous approximations of piecewise smooth solutions having a finite number of non-interacting shocks and established a convergence rate \(O(1)\varepsilon^\gamma\) for any \(\gamma < 1\). Bressan and Yang [11] considered the case of general BV solutions to hyperbolic systems of conservation laws possibly with everywhere dense set of shocks and established a rate of convergence of the order

\[\|u^\varepsilon(\tau, \cdot) - u(\tau, \cdot)\|_{L^1} = O(1)(1 + \tau)\sqrt{\varepsilon} \ln \varepsilon|TV\{\bar{u}\}.\] (1.6)

We remark, that the Landau notation \(O\) denotes a quantity whose absolute value remains uniformly bounded.

In this paper, we obtain the convergence rate of (1.5) for the general case of system of balance laws, \(g \neq 0\) under the dissipativeness hypothesis (1.3). As in [11], the assumptions on the flux are strict hyperbolicity and genuine nonlinearity of characteristic fields. The main result is the following:

**Theorem 1.2.** Let system (1.1) be strictly hyperbolic and assume that each characteristic field is genuinely nonlinear and the dissipativeness hypothesis (1.3) holds. Then, given any initial data \(u(0, \cdot) = \bar{u}\) with small total variation and \(\bar{u} - u^* \in L^1\), there exists a positive constant \(\beta\), such that for every \(\tau > 0\), the solutions \(u\) and \(u^\varepsilon\) to (1.1)-(1.2) and (1.4), (1.2), respectively, satisfy the following estimate

\[\|u^\varepsilon(\tau, \cdot) - u(\tau, \cdot)\|_{L^1} = O(1)(1 + \tau)\sqrt{\varepsilon} \ln \varepsilon|TV\{\bar{u}\} + O(1)\varepsilon\|Dg\|_{\infty}(\|\bar{u} - u^*\|_{L^1} + \sqrt{\varepsilon}TV\{\bar{u}\})\] (1.7)
Remark 1.3. It should be noted that the constant $\beta$ is induced by the source $g$ and, in fact, $0 < \beta < \beta$. Moreover, as $g \to 0$ while (1.3) holds, the convergence rate (1.7) converges to bound (1.6) as obtained by Bressan and Yang in [11] for the case of conservation laws. Note the contribution of the source in the second term above to show that this term vanishes in the limit. Also, the term $e^{-\beta \tau}$ in the first term could have been omitted, but it is left there to emphasize that such a term appears because of the source.

It is worth pointing out the presence of the term $\frac{1-e^{-\beta \tau}}{\beta}$ in the right-hand side of (1.7). The reader should contrast (1.7) with the corresponding estimate (1.6) obtained in [11] for hyperbolic systems of conservation laws. Observe that by taking $\tau \to \infty$ in (1.7), the solution $u$ and the viscous approximations $u^\varepsilon$ both tend to the constant equilibrium $u^*$. Thus, the left hand side of (1.7) tends to zero. This result holds true because of the dissipative mechanism induced by $g$. This property is not expected for systems of conservation laws as indicated in (1.6), see [11]. In other words, a “better” convergence rate is expected to hold for systems of balance laws and the term $\frac{1-e^{-\beta \tau}}{\beta}$ clearly provides this.

Remark 1.4. It should be emphasized that the condition of diagonally dominance (1.3) even though may appear too restrictive to encompass many interesting physical examples of systems of balance laws with dissipative source such as the case of sources of the type commonly induced by relaxation, it can, in fact, accommodate systems with merely weakly dissipative sources for which the equilibrium state happens to be $L^1$-stable. Such systems have been treated successfully by Dafermos in [16], where the existence of global solutions is established under considerably weaker conditions than the diagonally dominance hypothesis (1.3) on the matrix $B$. In particular, for hyperbolic balance laws (1.1) endowed with a rich family of entropies, the dissipative properties of $g$ in the vicinity of the equilibrium state $u^*$ are encoded in the matrix

$$B(u^*) = R^{-1}(u^*)Dg(u^*)R(u^*),$$

where $R(u) = [r_1(u), \ldots, r_n(u)]$ is a matrix with columns the right eigenvectors of $Df(u)$. The matrix $B$ is weakly dissipative in the following sense:

- The equilibrium state $u^*$ is $L^1$-stable.
- The principal diagonal entries of $B$ are all positive:

$$B_{ii}(u^*) > 0, \ i = 1, \ldots, n.$$

Under an appropriate change of variables, the analysis in [16] shows that a system with a merely weakly dissipative source, in the above sense, can be reduced to a system that satisfies the diagonally dominance condition (1.3). The reduced system is of the form

$$W_t + F(W, P, Q)_x + G(W, P, Q) = 0,$$

where $P$ and $Q$ are known functions of $U$ and the new unknown variable is $W$. It should be noted that the source $G(\cdot, P, Q)$ satisfies the diagonal dominance condition. Since this system is not of the same homogeneous form as in (1.1), additional analysis is required in order to apply the analysis of the present article to (1.8). It is expected, however, that the analysis can be extended to include systems of this type and estimate (1.7) holds true also in that case for some appropriate constant $\beta > 0$. 


Having obtained this estimate, additional work is required in order to establish the corresponding estimate of the convergence rate of vanishing viscosity approximations for the original system. Following this process, our result on the convergence rate of vanishing viscosity approximations can be extended to systems with a merely weakly dissipative source. Note that this class of systems includes balance laws with frictional damping arising in elasticity.

Our analysis builds on the techniques introduced by Bressan and Yang in [11] for systems of conservation laws. More precisely, it relies on error estimates used to measure the interaction of waves. Shock waves are treated by monitoring the evolution of appropriate weighted Glimm-type functionals, while rarefaction waves are handled by employing a sharp decay estimate that measures the spreading of rarefaction waves [14].

It should be noted that results on the rate of convergence of various approximating procedures are of great interest in the context of the numerical investigation of solutions.

For the sake of completeness, we remark that the existence of solutions is also established for other classes of systems of balance laws. For example, if the inhomogeneity is present in the flux and the source, then rapid decay of the inhomogeneity and the source term as \( t \to \infty \) (cf. Dafermos–Hsiao [17]) or as \( |x| \to \infty \) in conjunction with nonzero characteristic speeds (cf. T. P. Liu [25]) is enough to guarantee global BV solution. Global existence of large solutions of bounded variation to a balance law for the evolution of granular flow has been obtained by Amadori and Shen in [3]. The system under consideration in [3] is “almost” of Temple class because of the linearly degeneracy of one of the two fields.

The outline of this article is as follows: In Section 2, the main strategy of our program is presented, starting with the construction of a suitable approximation \( v \) of the viscous solution \( u^\varepsilon \). In Section 3, wave measures and interaction functionals are introduced and their properties are discussed. These functionals play a crucial role in the proof of important error estimates which provide a measure of the interaction of waves. In Section 4, the proof of the main result is presented. In Section 5, estimates on the rarefaction waves are derived, starting with the investigation of the solution to the Burgers’s equation with impulsive source term and its comparison to the solution of the Burgers’s equation consisting only of one single rarefaction wave. In Section 6, estimates on shock fronts are established with the aid of properties of suitable weighted Glimm-type functionals. In the Appendix A, the modified front-tracking algorithm and the operator splitting method are presented for the sake of completeness.

2. Strategy. In this section, we present the main strategy of our approach. We first outline the major steps of the proof and then, apply them to the special case of a Lipschitz continuous solution \( u \) to (1.1) as a motivation in Subsection 2.1. In view of our analysis, an approximate function \( v \) is needed for the general case of a solution in BV and this is constructed in Subsection 2.2.

Let \( \varepsilon > 0 \) be fixed. Using the modified front-tracking method, for a small parameter \( \varepsilon' \) and a time step \( s = \Delta t \) small enough, one can construct a sequence of approximate solutions \( u^{\varepsilon',s} \) to (1.1) that are piecewise constant with finitely many discontinuities that satisfy

\[
\| u^{\varepsilon',s}(t) - u(t) \|_{L^1} < \varepsilon' + s
\]  

and such that all non-physical fronts have strength less than \( \varepsilon' > 0 \) and the operator splitting mechanism is applied at times \( t_k^s = k s, k = 1, \ldots, m \). We refer the reader to
Appendix A, for an outline of the modified front-tracking method. Now, we claim that without loss of generality, we can prove (1.7) with $u$ denoting a modified front-tracking approximate solution and not necessarily an exact solution of (1.1). Indeed, since we can take $\varepsilon', s << \varepsilon$, then (2.1) implies that any additional error terms that may appear in the difference (1.7) would be of the order of $O(1)(\varepsilon' + s)$ and thus, absorbed by the right-hand side of (1.7). From here and on, we call $u$ the $(\varepsilon', s)$-approximate solution constructed with the aid of a modified front-tracking approximation and also assume that the initial data $\bar{u}$ is piecewise constant.

The main steps of the proof of Theorem 1.2 are outlined below.

**Step 1.** Starting from $u$, we construct an approximation $v = v(x,t)$ of the viscous solution $u_\varepsilon$ for each fixed $\varepsilon$. First, by appropriately mollifying $u$ with respect to the space variable $x$, we get

$$u * \psi_{\sqrt{\varepsilon}} = \int u(t,y)\psi_{\sqrt{\varepsilon}}(x-y)dy,$$

and subsequently, by inserting suitable viscous shock profiles at the locations of finitely many big shocks, we define

$$v = u * \psi_\delta + \sum_{p \in BS} (\chi_p - \vartheta_p).$$  \hspace{2cm} (2.2)

Here, $\delta = \sqrt{\varepsilon}$ and we use the rescaling function

$$\psi_\delta := \frac{1}{\delta} \psi(\frac{x}{\delta}),$$

where $\psi : \mathbb{R} \to [0,1]$ denotes a smooth even function satisfying the following properties

$$\psi(s) = 0 \quad \text{if} \quad |s| > \frac{2}{3},$$

$$s \psi'(s) \leq 0, \quad \text{for all} \quad s,$$  \hspace{2cm} (2.3)

and $\int \psi(s)ds = 1$. The functions $\chi_p$ and $\vartheta_p$ are to be defined in Subsection 2.2 as well as the class of waves $p \in BS$.

For each time $t^+_k$, $k = 1, \ldots, m$ at which the operator splitting method applies, we consider the set $\{t^+_1, \ldots t^+_m\}$ of interaction times in the modified front tracking solution $u$ and we denote by $\delta_0 = TV(\bar{u})$ the total variation of the initial data $\bar{u}$ and $\delta_1 = \|\bar{u} - u^*\|_{L^1}$. Then, the approximation $v = v(t,x)$ satisfies the following four properties:

1. The function $v$ is discontinuous at the time discretization $t = t^+_k$, $k = 1, \ldots, m$ of the operator splitting method and, for each $k$, at the interaction times $t = t^+_i$, $i = 1, \ldots, N_k$.

2. \begin{equation}
\|v(0) - \bar{u}\|_{L^1} = O(1) \cdot \delta_0 \sqrt{\varepsilon}, \quad \|v(\tau) - u(\tau)\|_{L^1} = O(1) \cdot \delta_0 \sqrt{\varepsilon} e^{-\beta \tau}. \hspace{2cm} (2.4)
\end{equation}

3. \begin{equation}
\sum_{k=1}^m \sum_{1 \leq i \leq N_k} \int |v(t^+_k, x) - v(t^+_i, x)|dx = O(1) \cdot \delta_0 \sqrt{\varepsilon} |\ln \varepsilon|. \hspace{2cm} (2.5)
\end{equation}

4. \begin{equation}
\sum_{k=1}^m \int |v(t^+_k, x) - v(t^-_k, x)|dx = O(1) \cdot \|Dg\|_\infty (\delta_1 \varepsilon + \delta_0 \varepsilon \sqrt{\varepsilon}). \hspace{2cm} (2.6)
\end{equation}
4.

\[ \int_0^\tau \int |v_t + A(v)v_x + g(v) - \varepsilon v_{xx}| \, dx \, dt = O(1) \cdot \left( 1 + \frac{1 - e^{-\beta \tau}}{\beta} \right) \delta_0 \sqrt{\varepsilon} |\ln \varepsilon|. \]

(2.7)

The construction of \( v \) is given in Subsection 2.2 and the above properties follow from the analysis in the following sections. Moreover, as it shown in Subsection 2.1, the mollification term in (2.2) is not sufficient to provide an approximation function that would satisfy the above four properties. Hence, this raises the need to insert viscous shock profiles at points, where there are shocks of appropriate big strength in order to achieve a sharper approximation \( v \). This is reflected in the second term of the expression (2.2).

**Step 2.** In this step, we estimate Property 4 for the case of rarefaction fronts. First, we study a solution that consists of a single centered rarefaction wave and then, we deal with the presence of a large number of centered rarefaction waves. By employing a series of comparison lemmas and introducing appropriate functionals, we treat the general case. Indeed, we prove that the integral in (2.7) for solutions consisting of a large number of centered rarefaction waves is of the same order as the one for the case of a single centered rarefaction wave. The comparison lemmas are established in Section 5 as a consequence of a sharp decay estimate for positive waves obtained in the context of hyperbolic balance laws in [14] and for hyperbolic conservation laws in [10].

**Step 3.** Next, we prove Property 4 for the case of shock fronts. More precisely, we establish error estimates by introducing new Lyapunov functionals that control interactions of shock waves in the same family and also interactions of waves in different families. The analysis is presented in Section 6.

**Step 4.** Combining the results from Steps 2 and 3 and analyzing carefully the interaction of waves at all times \( t_k \), we show that the approximate function \( v \) satisfies Properties 1–4 given in Step 1. This is established in Section 4.

**Step 5.** Last, we employ an error estimate and Properties 1–4 to derive the convergence rate (1.7) as follows:

We first recall that if \( \mathcal{L} \) is a Lipschitz continuous semigroup on a domain \( \mathcal{D} \), then the following estimate holds:

\[ \mathcal{L} : \mathcal{D} \times [0, \infty) \rightarrow \mathcal{D}, \]

\[ \| \mathcal{L}_t w(0) - w(t) \|_{L^1} \leq L \int_0^t \liminf_{h \to 0^+} \frac{\| \mathcal{L}_h w(\tau) - w(\tau + h) \|_{L^1}}{h} \, d\tau, \]

(2.8)

where \( L \) is the Lipschitz constant of the semigroup and \( w(\tau) \in \mathcal{D} \). The error estimate (2.8) appears extensively in the theory of front tracking method (cf. Bressan [5]). In fact, this is one of the main ingredients in establishing (a) that, if the semigroup \( \mathcal{L} \) associated with system (1.1) exists, then the entropy weak solution obtained by the front tracking method coincides with the trajectory of the semigroup \( \mathcal{L} \) and (b) uniqueness within the class of viscosity solutions. We refer the reader to Appendix A and the references therein.

Now, we consider the semigroup \( \mathcal{L}^\varepsilon \) generated by the parabolic system (1.4). As proven in [12], it is Lipschitz continuous with respect to the initial data, namely there
exists a constant $L > 0$ such that
\[ \| \mathcal{L}_t^\varepsilon \bar{u} - \mathcal{L}_t^\varepsilon \bar{v} \|_{L^1} \leq Le^{-\beta t} \| \bar{u} - \bar{v} \|_{L^1}, \quad \forall t. \]

It remains to estimate the $L^1$-distance between the solution $u$ of the balance law (1.1) and the solution $u^\varepsilon$ of its viscous approximation (1.4). Using the semigroup $\mathcal{L}_t^\varepsilon$, the approximation $v$, and error estimate (2.8), we get
\begin{align}
\| u^\varepsilon(\tau) - u(\tau) \|_{L^1} &\leq \| \mathcal{L}_t^\varepsilon \bar{u} - \mathcal{L}_t^\varepsilon \bar{v}(0) \|_{L^1} + \| \mathcal{L}_t^\varepsilon \bar{v}(0) - v(\tau) \|_{L^1} + \| v(\tau) - u(\tau) \|_{L^1} \\
&\leq Le^{-\beta \tau} \| \bar{u} - v(0) \|_{L^1} + \| \mathcal{L}_t^\varepsilon \bar{v}(0) - v(\tau) \|_{L^1} + \| v(\tau) - u(\tau) \|_{L^1} \\
&\leq Le^{-\beta \tau} \| \bar{u} - v(0) \|_{L^1} + L \int_0^\tau \int |v_t + A(v)v_x + g(v) - \varepsilon v_{xx}| \, dx \, dt \\
&\quad + L \sum_{k=1}^m \sum_{1 \leq i \leq N_k} \| v(t^k_i+) - v(t^k_i-) \|_{L^1} \\
&\quad + L \sum_{k=1}^m \| v(t^k_i+) - v(t^k_i-) \|_{L^1} + \| v(\tau) - u(\tau) \|_{L^1}. \tag{2.9}
\end{align}

By Properties 1–4, the result of Theorem 1.2 follows immediately.

In view of the above analysis, the aim is to construct the approximate function $v$ in such a way so that Properties 1–4 hold true. As a motivation of our strategy, we derive in Subsection 2.1 the convergence rate
\[ \| u^\varepsilon(\tau) - u(\tau) \|_{L^1} = \mathcal{O}(1) \cdot \delta_0 \sqrt{\varepsilon} \left( e^{-\beta \tau} + \frac{1 - e^{-\beta \tau}}{\beta} \right) \tag{2.10} \]
for the case of a Lipschitz continuous solution $u$ to (1.1). It should be noted that the logarithmic factor is missing in the above bound. This term appears in the presence of rarefaction waves and arises by carefully monitoring the decay rate of these waves. The general case for which the solution is of bounded variation and therefore, it may admit countably many discontinuities requires the analysis of Steps 2–4 and this is given in the following sections.

### 2.1. Motivation

We assume that system (1.1) is strictly hyperbolic and satisfies the dissipativeness hypothesis (1.3). Then, if $\delta_0 = \text{TV} \{ \bar{u} \}$, is sufficiently small, the total variation of the solution is uniformly bounded for all times and exponentially decaying in time, i.e.
\[ \text{TV} \{ u(t) \} = \mathcal{O}(1) \cdot \delta_0 e^{-\beta t}, \quad \| u^\varepsilon_x \|_{L^1} = \mathcal{O}(1) \cdot \delta_0 e^{-\beta t} \text{ for all } t \geq 0, \tag{2.11} \]
where $\beta$ is a positive constant induced by the dissipation mechanism. See Christoforou [12, 13].

Now, recalling the function $\psi$ in (2.3) and letting $v^\delta(t) \doteq u(t) * \psi_\delta$, we have that
\begin{align}
\| v^\delta(t) - u(t) \|_{L^1} &= \int | \int (u(t,x) - u(t,y)) \psi_\delta(x-y) \, dy | \, dx \\
&\leq \int \text{Tot. Var.} \{ u; [x - \delta, x + \delta] \} \, dx \\
&= \mathcal{O}(1) \cdot \delta_0 \delta e^{-\beta t}. \tag{2.12}
\end{align}
For the sake of motivation, let us assume here that $u$ is absolutely continuous. Then, we estimate the terms on the right-hand side of (2.9) by taking as an approximation $v$ the smooth mollification $v^\delta$. Hence, we obtain

\[
\int |\varepsilon v^\delta_{xx}(x)|dx = \varepsilon \int |(u_x * \psi_{\delta,x})(x)|dx \\
\leq \varepsilon \|u_x\|_{L^1} \cdot \|\psi_{\delta,x}\|_{L^1} \\
= O(1) \cdot \delta_0 \varepsilon e^{-\beta t},
\]

equation (2.13)

and

\[
\int |v^\delta_t + A(v^\delta)v^\delta_x + g(v^\delta)|dx \leq \int \left| \left( \int A(v^\delta(x))u_x(y) - A(u(y))u_x(y) \right) \psi_{\delta}(x-y)dy \right| dx \\
+ \int \left| \left( \int g(v^\delta(x)) - g(u(y)) \psi_{\delta}(x-y)dy \right) dx \right| \\
\leq \int \left( \int |A(u^\delta(x)) - A(u(y))| \psi_{\delta}(x-y)dx \right) |u_x(y)|dy \\
+ \int \left( \int |g(v^\delta(x)) - g(u(y))| \psi_{\delta}(x-y)dy \right) dx \\
\leq O(1) \cdot \|DA\|_{C^0} \int \text{Osc}\{u; [y - \delta, y + \delta]\} |u_x(y)|dy \\
+ O(1) \cdot \|Dg\|_{C^0} e^{-\beta t} \delta_0 \delta,
\]
equation (2.14)

where $\text{Osc}\{u; [a, b]\}$ stands for the oscillation of the function $u$ over the interval $[a, b]$.

By the Lipschitz continuity property, the oscillation of $u$ on any interval of length $2\delta$ is $O(1) \cdot \delta$. Hence, the above analysis yields

\[
\int |v^\delta_t + A(v^\delta)v^\delta_x + g(v^\delta)|dx \leq O(1) \delta_0 e^{-\beta t}.
\]
equation (2.15)

Combining (2.11)–(2.15) with (2.9), we arrive at

\[
\|v^\varepsilon(\tau) - u(\tau)\|_{L^1} \leq \|L^\varepsilon \tilde{u} - v^\delta(\tau)\|_{L^1} + \|v^\delta(\tau) - u(\tau)\|_{L^1} \\
\leq Le^{-\beta \varepsilon} \|	ilde{u} - v^\delta(0)\|_{L^1} + L \int_0^\tau \int |v^\delta_t + A(v^\delta)v^\delta_x + g(v^\delta) - \varepsilon v^\delta_{xx}|dxdt \\
+ \|v^\delta(\tau) - u(\tau)\|_{L^1} \\
= O(1) \cdot \delta_0 \sqrt{\varepsilon} \left( e^{-\beta \varepsilon} + \frac{1 - e^{-\beta \tau}}{\beta} \right),
\]
equation (2.16)

for $\delta := \sqrt{\varepsilon}$. Thus, the convergence rate (1.7) in the theorem for the case of a Lipschitz continuous solution $u$ holds true.

In general, the solution $u$ is not Lipschitz continuous but a function of bounded variation possibly with a countable number of shocks. So the above estimate does not hold true in the general case. To treat the general case, the mollification $v^\delta$ is not adequate approximation of the solution $u$. Indeed, there are fronts in $u$ that result to additional error terms in (2.14). For genuinely nonlinear systems, the additional error terms due to centered rarefaction waves can be controlled by carefully estimating the decay rate of these waves. This analysis is presented in Section 5. A sharp decay rate
on positive waves for hyperbolic balance laws (1.1) under the dissipativeness hypothesis (1.3) has been established by Christoforou and Trivisa in [14]. This estimate is employed in Section 5 for the control of the Lyapunov functionals corresponding to the presence of rarefaction waves in the solution $u$. However, there are also shock fronts of large strength that cannot be controlled using only the mollification $v^\delta$. Thus, a sharper approximation $v$ that would cancel the effect of those fronts is needed. The construction of the approximation $v$ is given in the next subsection.

2.2. Construction of approximate function $v$. We assume that system (1.1) is strictly hyperbolic and all characteristic fields are genuine nonlinear. Also, we recall that $u$ denotes a modified front tracking approximate solution to system (1.1).

For each $i = 1, \ldots, n$, we measure the (signed) strength of an $i$-front connecting the states $u^-$, $u^+$ as

$$\sigma := \lambda_i(u^+) - \lambda_i(u^-).$$

Here, we construct a function $v$ that is piecewise smooth and approximates $u$ satisfying Properties 1–4 given in Step 1 of Section 2.

Given $\vartheta > 0$, one can select finitely many shocks with the following properties [6]:

For each $t$ in the time interval $I_p = (t_p^-, t_p^+)$, $p = 1, \ldots, \kappa = O(1) \cdot \vartheta_0^{1/\vartheta}$, except possibly for a finite number of interaction points, the states $u^-_p$ and $u^+_p$ are connected by a shock $t \mapsto x_p(t)$ of family $k_p$, with strength

$$|\sigma_p| \geq \vartheta,$$

while

$$|\sigma_p(t^*)| \geq \vartheta, \text{ for some } t^* \in I_p.$$

It should be noted that every shock in the solution constructed by the modified front tracking with strength greater than $\vartheta$ is part of the above fronts. All shock fronts that satisfy the above two properties are called big shocks and we denote this class of fronts by $BS$.

Moreover, for each $p$ and each $t \in I_p$, except possibly a finite number of interaction points, we consider the viscous shock profile $\omega_p$, connecting the states $u^-_p$ and $u^+_p$, namely, the solution to

$$\omega''_p = (A(\omega_p) - \lambda_p)\omega'_p + \varepsilon g(\omega_p), \quad \lim_{s \to \pm \infty} \omega_p(r) = u^\pm_p,$$

where $\lambda_p$ denotes the shock speed. Without loss of generality, we choose the parameter $r$ in such a way so that $r = 0$ corresponds to the center of the traveling profile. Then, by rescaling, we consider the shock profile

$$r \to \omega^\varepsilon_p(r) := \omega_p \left( \frac{r}{\varepsilon} \right)$$

to the viscous approximation (1.4).

Next, using an appropriate transformation from $\mathbb{R}$ to the open interval

$$O_p(t) := (x_p(t) - \delta, x_p(t) + \delta),$$
we define a viscous shock profile $\chi_p$ by setting

$$
\chi_p(x_p + \zeta) := \begin{cases} 
\omega^\varepsilon_p(\theta(\zeta)) & \text{if } \zeta \in (-\delta, \delta) \\
u^+_p & \text{if } \zeta \geq \delta \\
u^-_p & \text{if } \zeta \leq -\delta,
\end{cases}
$$

(2.17)

where

$$
\theta(\zeta) = \begin{cases} 
\zeta & \text{if } |\zeta| \leq \frac{\sqrt{\varepsilon}}{2} \\
\frac{\varepsilon}{4(\sqrt{\varepsilon} - \zeta)} & \text{if } \frac{\sqrt{\varepsilon}}{2} \leq \zeta \leq \sqrt{\varepsilon} \\
-\frac{\varepsilon}{4(\sqrt{\varepsilon} + \zeta)} & \text{if } -\sqrt{\varepsilon} \leq \zeta \leq -\frac{\sqrt{\varepsilon}}{2}.
\end{cases}
$$

(2.18)

Also, for $p \in BS$, we introduce the function $\vartheta_p$

$$
\vartheta_p(x_p + \zeta) := u^+_p \int^\zeta \psi_{\delta}(y) dy + u^-_p \int^{+\infty}_{\zeta} \psi_{\delta}(y) dy,
$$

(2.19)

where the cutoff $\psi_{\delta}$ is given in (2.3), in order to replace the mollification $u \ast \psi_{\delta}$ by the viscous shock profile $\chi_p$ over $O_p(t)$ for the case of $p \in BS$.

Finally, we define

$$
v = u \ast \psi_{\delta} + \sum_{p \in BS} (\chi_p - \vartheta_p),
$$

where the summation ranges over all big shock fronts. In the next sections, we verify that by choosing

$$
\delta := \sqrt{\varepsilon}, \quad \vartheta := 4\sqrt{\varepsilon} |\ln \varepsilon|,
$$

Properties 1–4 of $v$ and, thus, the main result of this article (1.7) hold true.

3. Preliminaries. In this section, we give some preliminaries on wave measures and interaction functionals. More precisely, in Subsection 3.1, the strength of the $i$-wave of a function of small total variation is defined as well as a partial ordering on positive Radon measures. The latter is used primarily when comparing the solution to (1.1) with the one of Burgers’s equation with impulsive source terms for the case of rarefaction fronts. We refer to the result in [14] as stated in Section 5. In Subsection 3.2, we define Lyapunov functionals that are mostly needed to control terms due to interaction of waves for the case of shock fronts.

3.1. Measures. Here and in what follows, $u(t, \cdot)$ denotes a function of small total variation, whose distributional derivative $D_x u$ is a Radon measure. Following [5] and [10], we define the wave measures $\mu^i$ of $i$-waves in $u$ as follows:

$$
\mu^i = l_i(u) \cdot D_x u
$$
on the sets of continuity of $u$ and

$$
\mu^i(\{x\}) = \sigma_i
$$
at the points of jump of $u$. Here, $\sigma_i$ denotes the strength of the $i$-wave in the Riemann solution with data the left $u_l = u(x-)$ and right $u_r = u(x+)$ states. If we denote by
\[ u_i = u_0, u_1, \ldots, u_n = u_r \] are the states in the Riemann solution, and take into consideration the fact that all characteristic fields are genuine nonlinear, then we can define the strength of the \( i \)-wave to be

\[ \sigma_i = \lambda_i(u_i) - \lambda_i(u_{i-1}). \]

Moreover, we denote by \( \mu^+ \) and \( \mu^- \) the positive and negative parts of \( \mu \).

**Definition 3.1.** Given two positive Radon measures \( \mu, \mu' \), we say that \( \mu \lesssim \mu' \) if and only if

\[ \sup_{|A| \leq s} \mu(A) \leq \sup_{|B| \leq s} \mu'(B), \]

for every \( s > 0 \). Here, \( |A| \) denotes the Lebesgue measure of the set \( A \).

We remark that the usual relation \( \mu \leq \mu' \) if and only if \( \mu(A) \leq \mu'(A) \) for every \( A \subset \mathbb{R} \), is stronger, in the sense that it implies \( \mu \lesssim \mu' \), but the reverse does not hold.

The ordering relation \( \lesssim \) can also be interpreted in terms of rearrangements. Let \( v \) be a bounded, nondecreasing function \( v: \mathbb{R} \mapsto \mathbb{R} \) and \( \mu = D_x v \) its distributional derivative. A positive Radon measure \( \mu \) can be decomposed as the sum of a singular and an absolutely continuous part with respect to the Lebesgue measure, namely

\[ \mu = \mu^{\text{sing}} + \mu^{\text{ac}}. \]

The absolutely continuous part corresponds to the usual derivative \( z = v_x \), which is a nonnegative \( L^1 \) function defined almost everywhere.

**Definition 3.2.** The symmetric rearrangement of \( z \), denoted by \( \tilde{z} \) is defined to be the unique even function with the properties

\[ \tilde{z}(x) = \tilde{z}(-x), \quad \tilde{z}(x) \geq \tilde{z}(x') \quad \text{if} \quad 0 < x < x', \]

\[ \text{meas}\{x; \tilde{z}(x) > c\} = \text{meas}\{x; z(x) > c\} \quad \text{for every} \quad c > 0. \]

In accordance, the odd rearrangement of \( v \), denoted by \( \hat{v} \) is defined to be the unique function with the property

\[ \hat{v}(-x) = -\hat{v}(x), \quad \hat{v}(0+) = \frac{1}{2}\mu^{\text{sing}}(\mathbb{R}), \]

\[ \hat{v}(x) = \hat{v}(0+) + \int_0^x z(y) dy \quad \text{for} \quad x > 0. \]

Hence \( \hat{v} \) is convex for \( x < 0 \) and concave for \( x > 0 \).

Using the odd rearrangement \( \hat{v} \) and the partial ordering \( \lesssim \) we have the following comparison lemma.

**Lemma 3.3.** Let \( u, v: \mathbb{R} \mapsto \mathbb{R} \) be two nondecreasing \( BV \) functions and \( \hat{u}, \hat{v} \) be the corresponding odd rearrangements. Then the following two properties hold:

(i) \( \int_{-\infty}^{\infty} \text{Tot.Var.}\{u; [x-\rho, x+\rho]\} du(x) \leq 3 \int_{-\infty}^{\infty} [\hat{u}(x+\rho) - \hat{u}(x-\rho)] d\hat{u}(x). \)
(ii) If $Dxu \preceq Dxv$, then
\[
\int_{-\infty}^{\infty} [\hat{u}(x + \rho) - \hat{u}(x - \rho)] d\hat{u}(x) \leq \int_{-\infty}^{\infty} [\hat{v}(x + \rho) - \hat{v}(x - \rho)] d\hat{v}(x).
\]

For more details on symmetric and odd rearrangements, we refer the reader to the book [5] of Bressan and the proof of the above lemma can be found in [11], pages 1084–1086.

Last, in the context of measures $\mu^i$, we define the Glimm functionals
\[
V(u) = \sum_i |\mu_i|(\mathbb{R}), \tag{3.1}
\]
\[
Q(u) = \sum_{i<j} (|\mu^i| \otimes |\mu^j|)\{(x, y); x < y\} + \sum_i (\mu^i - \otimes |\mu^i|)\{(x, y); x \neq y\}. \tag{3.2}
\]

3.2. Functionals. Following the notation in [7, 24], we rewrite the Glimm functionals (3.1) and (3.2) as follows: the total strength of waves
\[
V(u) = \sum_p |\sigma_p|, \tag{3.3}
\]
and the interaction potential
\[
Q(u) := \sum_{(p,q) \in \mathcal{A}} |\sigma_p \sigma_q|, \tag{3.4}
\]
in the modified front tracking solution $u$. Here, the second summation ranges over the set $\mathcal{A}$ of all couples of approaching wave fronts at least one being a shock. It is easy to check that functionals (3.1) and (3.2) are equivalent to (3.3) and (3.4), respectively. Throughout this article, we use expressions (3.3) and (3.4) unless otherwise is noted. Also, the functional $\Upsilon$ is given by
\[
\Upsilon(u) := V(u) + C_0 Q(u).
\]

Under the dissipativeness condition (1.3), if the total variation of $u$ is small and the constant $C_0$ is large enough, the functionals $Q(u(t))$ and $\Upsilon(u(t))$ are nonincreasing in time and more precisely, they are decaying exponentially fast. The decrease in $Q$ controls the amount of interaction, while the decrease in $\Upsilon$ controls both the interaction and the cancellation in the solution.

Next, we design a functional $\hat{Q}(u)$ such that the map $t \mapsto \hat{Q}(u(t))$ is non-increasing except at times where a new big shock is introduced. We recall that the definition of big shocks is given in Subsection 2.2. We construct the functional $\hat{Q}(u(t))$ in the same spirit as in [11], namely as a linear combination of the functionals $Q^l$, $Q^m$ $Q^r$:
\[
\hat{Q}(u(t)) := \sqrt{\varepsilon} |\ln \varepsilon| \cdot \left( C_1 \Upsilon(u) + C_2 Q^l(u) + C_3 Q^m(u) \right) + \sqrt{\varepsilon} Q^r(u), \tag{3.5}
\]
where $Q^l$, $Q^m$ and $Q^r$ are defined below. First, we define $Q^l$ to be
\[
Q^l(u) := \sum_{k_p \neq k_q} W_{pq}(x) |\sigma_p \sigma_q|, \tag{3.6}
\]
and the summation extends over couples of fronts of different families either shocks or rarefactions, then we take
\[ Q^m(u) := \sum_{p \in BS} W^m_p(x) D_x \tilde{w}_p, \tag{3.7} \]
to control of the interaction between big shocks and rarefactions of the same family, and, last,
\[ Q^r(u) := \sum_{p \in S} \left| \sigma_p \right| \int W^m_p(x) W^r_p(x) D_x \tilde{z}_p, \tag{3.8} \]
where the summation runs over all shock fronts of the same family. The definitions of \( W^l_{pq} \), \( W^m_p \), \( W^r_p \), \( \tilde{w}_p \) and \( \tilde{z}_p \) are given below. It should be noted that the weights of these three functionals are the same as introduced in [11].

* The weights \( W^l_{pq} \in [0, 1] \), in (3.6) are defined as follows:
  - If \( k_q < k_p \), then
    \[ W^l_{pq} := \begin{cases} 
    0, & \text{if } x_q < x_p - 2\sqrt{\varepsilon}, \\
    \frac{1}{2} + \frac{x_p - x_q}{4\sqrt{\varepsilon}}, & \text{if } x_q \in [x_p - 2\sqrt{\varepsilon}, x_p + 2\sqrt{\varepsilon}], \\
    1, & \text{if } x_q > x_p + 2\sqrt{\varepsilon}.
    \end{cases} \]
  - If \( k_q > k_p \), then
    \[ W^l_{pq} := \begin{cases} 
    1, & \text{if } x_q < x_p - 2\sqrt{\varepsilon}, \\
    \frac{1}{2} - \frac{x_q - x_p}{4\sqrt{\varepsilon}}, & \text{if } x_q \in [x_p - 2\sqrt{\varepsilon}, x_p + 2\sqrt{\varepsilon}], \\
    0, & \text{if } x_q > x_p + 2\sqrt{\varepsilon}.
    \end{cases} \]

* The weights \( W^m_p(x) \) in (3.7) are given by
  \[ W^m_p(x) := \min \left\{ \frac{1}{2} + \frac{|x - x_p|}{4\sqrt{\varepsilon}}, 1 \right\}, \]
whereas the function \( \tilde{w}_p \) is
  \[ \tilde{w}_p(x) := \begin{cases} 
    -\frac{|\sigma_p|}{4}, & \text{if } w_p(x) < -\frac{|\sigma_p|}{4}, \\
    w_p(x), & \text{if } |w_p(x)| \leq \frac{|\sigma_p|}{4}, \\
    \frac{\sigma_p}{4}, & \text{if } w_p(x) > \frac{|\sigma_p|}{4},
    \end{cases} \]
with
  \[ w_p(x) := \begin{cases} 
    \sum_{q \in R_p, x_q \in [x, x_p]} (-\sigma_q) & \text{if } x < x_p, \\
    \sum_{q \in R_p, x_q \in [x_p, x]} (\sigma_q) & \text{if } x > x_p.
    \end{cases} \]

Here \( \sigma_p \) stands for the strength of a big shock of the \( k_p \) family located at \( x_p \) and \( R_p \) the set of all rarefaction fronts of the same family \( k_p \).
* The weights $W^\tau_p(x)$ in (3.8) are given by
\[
W^\tau_p(x) := \begin{cases} 
(\varepsilon - z_p(x-))^{-1} & \text{if } x < x_p, \\
(\varepsilon - z_p(x+))^{-1} & \text{if } x > x_p,
\end{cases}
\]
while the function $\tilde{z}$ is
\[
\tilde{z}_p(x) = \begin{cases} 
\min\{z_p(x'); x < x' < x_p\} & \text{if } x < x_p, \\
\max\{z_p(x'); x_p < x' < x\} & \text{if } x > x_p,
\end{cases}
\]
with
\[
z_p(x) := \begin{cases} 
-\frac{\sigma_p}{2} - \sum_{q \in S_p, x < x_q < x_p} |\sigma_q| + \sum_{q \in R_p, x < x_q < x_p} 3\sigma_q & \text{if } x < x_p, \\
\frac{\sigma_p}{2} - \sum_{q \in S_p, x < x_q < x} |\sigma_q| - \sum_{q \in R_p, x_p < x_q < x} 3\sigma_q & \text{if } x > x_p.
\end{cases}
\]

Also, here, $\sigma_p$ stands for the strength of a shock front of the $k_p$ family located at $x_p$ and $S_p, R_p$ the sets of all shock and rarefaction fronts of the same family $k_p$, respectively.

The total increase of the functional $\tilde{Q}(u(t))$ at times where a big shock of strength $\sigma_p$ is created, is shown to be $O(1) \cdot \sqrt{\varepsilon}\ln \|\sigma_p\|$ in Section 6.

4. Proof of the main theorem. In this section, we prove the main result of the paper, estimate (1.7), following the steps outlined in Section 2.

Let $\varepsilon > 0$ be fixed. For a small parameter $\varepsilon' > 0$ and a sufficiently small time step $s = \Delta t$ within the operator splitting method, we denote by $u$ the $(\varepsilon', s)$-approximate solution constructed via the modified front-tracking method to system (1.1). As explained in Section 2, it suffices to prove estimate (1.7) for the approximation $u$ and the solution $u^\varepsilon$ to (1.4).

Having constructed the viscous shock profile $\chi_p$ centered at points $x_p$, where a big shock $p \in BS$ is present in the solution $u$ and the mollification $u^\delta = u * \psi_\delta$, we define the approximation
\[
v = u * \psi_\delta + \sum_{p \in BS} (\chi_p - \psi_p) 
\]
for $\delta = \sqrt{\varepsilon}$, $\theta = \sqrt{\varepsilon}\ln \varepsilon$ and $\psi_p$ defined in (2.19). In view of the analysis in Subsection 2.1, the mollification $u * \psi_\delta$ does not suffice to provide Property 4 at locations where $u$ is discontinuous, therefore we need to correct this approximation by inserting a modified viscous shock profile $\chi_p$ at points, where there is a big shock.

We recall that the goal is to establish Properties 1–4 of the approximation $v$ as given in Step 1 of Section 2. By construction of the function $v$, Properties 1 and 2 follow immediately.

To establish Property 3, we observe that within the time interval $[t_{k-1}^*, t_k^*]$, $k = 1, \ldots, m$, the approximation $v$ is discontinuous precisely at times $t_i^k$, $i = 1, \ldots, N_k$, when we have the occurrence of an interaction involving a big shock at a point $x_p$. Indeed, the change of the left and right states $u_p^- = u(t_i^k, x_p^-)$ and $u_p^+ = u(t_i^k, x_p^+)$ at those times $t_i^k$ and the definitions (2.18)–(2.19) imply that $|v(t_i^k +, x) - v(t_i^k -, x)| \neq 0$ for $x \in [x_p - \sqrt{\varepsilon}, x_p + \sqrt{\varepsilon}]$. To estimate (2.5), we investigate five cases that cover all possible interactions involving big shocks.
Hence, it follows that \( \sigma \) family is created, that is of strength these quadratic terms are controlled by the decrease of the interaction potential \( Q \) family of strength \( \sigma \) at \( x \) and the total contribution due to this type of interactions is \( O(1) \sqrt{\varepsilon} \delta_0 \).

CASE 2. Suppose that a big shock of strength \( \sigma_p \) is terminated at time \( t_i^k \). Then, as in the previous case,

\[
\int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} |v(t_i^k +, x) - v(t_i^k -, x)| \, dx = O(1) \sqrt{\varepsilon} |\sigma_p| ,
\]

and again, the total contribution due to this type of interactions is \( O(1) \sqrt{\varepsilon} \delta_0 \).

CASE 3. Suppose that a big shock \( p \in BS \) interacts with a front of different family of strength \( \sigma_q \) at \( x = x_p \). Then, \( |v(t_i^k +, x) - v(t_i^k -, x)| = O(1) |\sigma_p| |\sigma_q| \) and these quadratic terms are controlled by the decrease of the interaction potential \( Q(u) \). Hence, it follows that

\[
\int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} |v(t_i^k +, x) - v(t_i^k -, x)| \, dx = O(1) \sqrt{\varepsilon} |\sigma_p| |\sigma_q| ,
\]

and the total contribution due to this type of interactions is \( O(1) \sqrt{\varepsilon} (TV \tilde{u})^2 \).

CASE 4. Suppose that a big shock \( p \in BS \) interacts with a small front of the same family of strength \( \sigma_q \). Then, the strength of the new outgoing shock \( \sigma'_p \) satisfies \( |\sigma'_p - \sigma_p| \leq |\sigma_q| \), which implies

\[
\int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} |v(t_i^k +, x) - v(t_i^k -, x)| \, dx = O(1) \sqrt{\varepsilon} |\sigma_q| .
\]

As before, the total contribution is \( O(1) \sqrt{\varepsilon} \delta_0 \).

CASE 5. Suppose that two big shocks of the same family interact at time \( t_i^k \) at the point \( x = x_p \), each one of strength \( \sigma_p \) and \( \sigma_q \). Then, one big shock of the same family is created, that is of strength \( \sigma'_p \) and satisfies \( \max\{\sigma_p, \sigma_q\} \leq \sigma'_p \leq \sigma_p + \sigma_q \). Hence,

\[
\int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} |v(t_i^k +, x) - v(t_i^k -, x)| \, dx = O(1) \sqrt{\varepsilon} \min\{|\sigma_p|, |\sigma_q|\} .
\]

Now, the above terms are bounded by the decrease of the functional \( Q' \). Following Lemma 6.2 in Section 6, we note that the total contribution to \( \sum_{k,l} |v(t_i^k +) - v(t_i^k -)|_{L^1} \) is \( O(1) \sqrt{\varepsilon} |\ln \varepsilon| \delta_0 \). Indeed, in the proof of Lemma 6.2, one can see that the functional \( Q' \) decreases at this rate, see [11, Lemma 3.1].

Combining the above five cases, we arrive at

\[
\sum_k \sum_{1 \leq i \leq N_k} |v(t_i^k +) - v(t_i^k -)|_{L^1} = O(1) \sqrt{\varepsilon} |\ln \varepsilon| \delta_0 \quad (4.2)
\]
and estimate (2.5) in Property 3 follows.

Next, we observe that \( v \) is discontinuous at the time discretization \( t^*_k = ks \), \( k = 1, \ldots, m \), of the operator splitting method. We recall that

\[
u(t^*_k +, x) = u(t^*_k, x) - s g(u(t^*_k, x)), \quad x \in \mathbb{R}
\] (4.3)

and

\[
\|u(t) - u^*\|_{L^1} \leq L e^{-\beta t}\|\bar{u} - u^*\|, \quad t > 0,
\] (4.4)

where \( u^* \) is the equilibrium state associated with system (1.1) and \( L \) a positive constant. For more details, we refer the reader to Appendix A. To estimate the jump of \( v \) at the times \( t^*_k \), \( k = 1, \ldots, m \), we study the change of \( u(t^*_k, \cdot) \) away from big shocks (BS) and along the intervals \( O_j = (x^*_j, -\sqrt{\varepsilon}, x^*_j, +\sqrt{\varepsilon}) \), where a big shock \( p_j \) is present before the time \( t = t^*_k \) or a big shock \( p'_j \) is present after the time \( t = t^*_k \). Indeed, \( v \) is discontinuous at \( t = t^*_k \) away from BS because \( u(t^*_k, \cdot) \) is defined by relation (4.3) and along the intervals \( O_{p_j} = (x^*_j, -\sqrt{\varepsilon}, x^*_j, +\sqrt{\varepsilon}) \) because the viscous shock profiles may change. More precisely, for each \( k = 1, \ldots, m \), we consider the points \( x^*_j \) at the time of discretization \( t^*_k \) of the operator splitting method where either a big shock \( p_j \) \( \in \) BS emanating at an earlier time \( t^*_k \) impinges the line \( t = t^*_k \) or a big shock \( p'_j \) \( \in \) BS starting from the line \( t = t^*_k \) prolongs at a later time \( t^*_k \). Hence, we write

\[
\|v(t^*_k +) - v(t^*_k -)\|_{L^1} = \sum_j \int_{O_{p_j}} |v(t^*_k +, y) - v(t^*_k -, y)| dy
\]

\[+ \int_{\mathbb{R} \setminus \bigcup_j O_{p_j}} |v(t^*_k +, y) - v(t^*_k -, y)| dy.
\] (4.5)

It should be noted that along the interval \( O_{p_j} = (x^*_j, -\sqrt{\varepsilon}, x^*_j, +\sqrt{\varepsilon}) \), the incoming front \( p_j \) or the outgoing front \( p'_j \) or even both fronts are big shocks. In any case, the strengths \( \sigma_{p_j} \) of the incoming front and \( \sigma'_{p_j} \) of the outgoing front satisfy via (4.3) the following bound

\[
|\sigma'_{p_j} - \sigma_{p_j}| \leq \mathcal{O}(1)s Dg\|L^\infty|\sigma_{p_j}|.
\] (4.6)

Although there is no interaction of fronts, we use the terminology “incoming” and “outgoing”, to distinguish the fronts before and after the time discretization \( t^*_k \). By (4.6), it follows

\[
\int_{O_{p_j}} |v(t^*_k +, y) - v(t^*_k -, y)| dy = \mathcal{O}(1)s \sqrt{\varepsilon} Dg\|L^\infty|\sigma_{p_j}|,
\]

which yields

\[
\sum_j \int_{O_{p_j}} |v(t^*_k +, y) - v(t^*_k -, y)| dy \leq \mathcal{O}(1)\varepsilon \sqrt{\varepsilon} Dg\|L^\infty TVu(t^*_k -).
\]

The second term in (4.5) can be estimated as follows

\[
\int_{\mathbb{R} \setminus \bigcup_j O_{p_j}} |v(t^*_k +, y) - v(t^*_k -, y)| dy \leq \int | \int [u(t^*_k +, y) - u(t^*_k -, y)] \psi_{\bar{u}}(x - y) dy | dx
\]

\[\leq s\|g(u(t^*_k -))\|_{L^1}
\]

\[\leq \mathcal{O}(1) s\|Dg\|_{L^\infty} \|\bar{u} - u^*\|_{L^1} e^{-\beta t^*_k},
\] (4.7)
These estimates play central role in the derivation of Property 4, on rarefaction and shock fronts, that are established in the following two sections. \[ \| v(t_k^+) - v(t_k^-) \|_{L^1} \leq O(1) \| Dg \|_{\infty} \left( \varepsilon \sqrt{\varepsilon} TV \bar{u} + \varepsilon \| u^* \|_{L^1} \right) \sum_{k=1}^{m} e^{\beta t_k^+}, \quad (4.8) \]

and estimate (2.6) in Property 3 follows immediately. We note that the above term arises in the estimate of the convergence rate only because we deal with systems of balance laws and as \( g \to 0 \), this term vanishes. To emphasize this, we keep the factor \( \| Dg \|_{\infty} \) in bound (4.8).

Now, it remains to prove Property 4. Before, we proceed, we state the estimates on rarefaction and shock fronts, that are established in the following two sections.

These estimates play central role in the derivation of Property 4.

**Proposition 4.1.** Under the hypotheses of Theorem 1.2, the function \( v \) defined in (4.1) satisfies

\[ \sum_{i=1}^{n} \int_{0}^{\tau} \sum_{p, q \in \mathcal{R}_i, |x_p - x_q| \leq 8 \sqrt{\varepsilon}} |\sigma_p \sigma_q| dt = O(1) (\ln(2 + 1 - e^{-\beta t}) + |\ln \varepsilon|) \sqrt{\varepsilon} \cdot TV \{ \bar{u} \} \quad (4.9) \]

and

\[ \int_{0}^{\tau} \sum_{p \in S(t)} \int_{O_p} |v_t + A(v) v_x + g(v) - \varepsilon v_{xx}| dx dt \]

\[ = O(1) \left( \frac{1 - e^{-\beta t}}{\beta} \right) (\varepsilon (1 + |\ln \varepsilon|) + \sqrt{\varepsilon}) TV \{ \bar{u} \} \]

\[ + O(1) \int_{0}^{\tau} \left( \sum_{x_{q_1} - x_{q_2} \in O_p(t), |x_{q_1} - x_{q_2}| \leq 2 \sqrt{\varepsilon}} |\sigma_{q_1} \sigma_{q_2}| - \sum_{x_r \in O_p(t)} |\sigma_r|^2 \right) dt, \quad (4.10) \]

where \( \mathcal{R}_i \) is the set of rarefaction fronts of the \( i \)-family, \( \sigma_p \) denotes the strength of the front at \( x_p \), \( S(t) \) is the set of shock fronts at time \( t \) and \( O_p = (x_p - \sqrt{\varepsilon}, x_p + \sqrt{\varepsilon}) \).

**Proof.** Estimates (4.9) and (4.10) are proved in Sections 5 and 6 respectively. To establish (4.9), we use results on the spreading of rarefaction waves that allow us to compare the positive waves to system (1.1) with a centered rarefaction wave to the Burger's equation with source. On the other hand, estimate (4.10) is obtained via a careful analysis of a Lyapunov functional with appropriate weights.

To prove (2.7), we first recall the analysis in Section 2. In account of (2.13)–(2.14) and the construction in Subsection 2.2, we have

\[ \int_{0}^{\tau} \int |v_t + A(v) v_x + g(v) - \varepsilon v_{xx}| dx dt = \int_{0}^{\tau} \sum_{p \in BS} E_p(t) dt + O(1) \frac{1 - e^{-\beta t}}{\beta} \sqrt{\varepsilon} TV \{ \bar{u} \} \]

\[ + O(1) \int_{0}^{\tau} \sum_{p \in BS} \int_{y \notin O_p} \text{Osc}\{u; [y - \delta, y + \delta]\} |du(y)| dy dt, \]

(4.11)

where \( O_p = (x_p - \sqrt{\varepsilon}, x_p + \sqrt{\varepsilon}) \) for a big shock \( p \in BS \) and

\[ E_p(t) = \int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} |v_t + A(v) v_x + g(v) - \varepsilon v_{xx}| dx dt . \]
It should be noted that, by definition, the function $v$ involves a modified viscous shock profile in each neighborhood of a big shock. Therefore, there are no terms in the above expression that describe interactions between big shocks. By (4.10) in Proposition 4.1, (4.11) reduces to

\[
\int_0^r \int |v_t + A(v)v_x + g(v) - \varepsilon v_{xx}| \, dx \, dt = O(1) \int_0^r \left( \sum_{|x_p - x_q| \leq 2\sqrt{\varepsilon}} |\sigma_p \sigma_q| - \sum_{r \in BS} |\sigma_r|^2 \right) dt \\
+ O(1) \cdot \frac{(1 - e^{-\beta \tau})}{\beta} \left( \varepsilon (1 + |\ln \varepsilon|) + \sqrt{\varepsilon} \right) TV\{\bar{u}\}.
\]

(4.12)

Next, we claim that for $t \in (t_k^*, t_{k+1}^*)$, $k = 0, \ldots, m - 1$, but away from interaction times, the following estimate holds

\[
\sum_{|x_p - x_q| \leq 2\sqrt{\varepsilon}} |\sigma_p \sigma_q| - \sum_{r \in BS} |\sigma_r|^2 = O(1) \cdot \sum_{i=1}^n \left( \sum_{p, q \in \mathcal{R}_i, |x_p - x_q| \leq 2\sqrt{\varepsilon}} |\sigma_p \sigma_q| \right) \\
+ O(1) \cdot \left| \frac{d}{dt} \hat{Q}(u(t)) \right| + O(1) \cdot \sqrt{\varepsilon} |\ln \varepsilon| e^{-\beta t} TV\{\bar{u}\}.
\]

(4.13)

and $\frac{d}{dt} \hat{Q}(u(t)) \leq 0$. One could prove this following the proof of Lemma 4.1 in [11], because within each time strip $(t_k^*, t_{k+1}^*)$ the solution $u$ constructed by the modified front-tracking algorithm is, roughly speaking, a solution to the homogeneous system with data $u(t_k^*, \cdot)$. Thus, the waves involved in the above expression can be treated exactly as in the case of systems of conservation laws. We refer the reader to Appendix A.2 for the algorithm and to Section 4 in Bressan–Yang [10] for the details of the proof of (4.13).

Using (4.9) in Proposition 4.1 and (6.8)–(6.9) in Lemma 6.2, we deduce

\[
\int_0^r \left[ \sum_{|x_p - x_q| \leq 2\sqrt{\varepsilon}} |\sigma_p \sigma_q| - \sum_{r \in BS} |\sigma_r|^2 \right] dt = \sum_{k=0}^{m-1} \int_{t_k^*}^{t_{k+1}^*} \left[ \sum_{|x_p - x_q| \leq 2\sqrt{\varepsilon}} |\sigma_p \sigma_q| - \sum_{r \in BS} |\sigma_r|^2 \right] dt \\
= O(1) \cdot \left[ 1 + \frac{1 - e^{-\beta \tau}}{\beta} \right] \sqrt{\varepsilon} |\ln \varepsilon| TV\{\bar{u}\}.
\]

(4.14)

Combining estimate (4.12) with (4.14), we arrive at

\[
\int_0^r \int |v_t + A(v)v_x + g(v) - \varepsilon v_{xx}| \, dx \, dt = O(1) \cdot \left[ 1 + \frac{1 - e^{-\beta \tau}}{\beta} \right] \sqrt{\varepsilon} |\ln \varepsilon| TV\{\bar{u}\}
\]

and Property 4 follows. In view of the analysis in Section 2 and using Properties 1–4,
we compute
\[
\|u^e(\tau) - u(\tau)\|_{L^1} \leq \|L^e_\tau \bar{u} - v(\tau)\|_{L^1} + \|v(\tau) - u(\tau)\|_{L^1} \\
\leq Le^{-\beta \tau} \|\bar{u} - v(0)\|_{L^1} + L \int_0^\tau \int |v_t + A(v)v_x + g(v) - \varepsilon v_{xx}|dxdt \\
+ L \sum_{k=1}^m \sum_{1 \leq i \leq N_k} \|v(t_i^k+) - v(t_i^k-)\|_{L^1} \\
+ L \sum_{k=1}^m \|v(t_i^k+) - v(t_i^k-\|_{L^1} + \|v(\tau) - u(\tau)\|_{L^1} \\
= \mathcal{O}(1) \left( 1 + e^{-\beta \tau} + \frac{1 - e^{-\beta \tau}}{\beta} \right) \sqrt{\varepsilon |\ln |TV\{\bar{u}\} | \\
+ \mathcal{O}(1) \varepsilon \|Dg\|_{\infty} (\|\bar{u} - u^*\|_{L^1} + \sqrt{\varepsilon |TV\{\bar{u}\}}),
\]
(4.15)
and the proof of Theorem 1.2 is now complete.

5. Estimates on Rarefaction Fronts. In this section, we prove estimate (4.9).

First, we state the result of Christoforou et al. in [14] that describes a sharp
decay estimate on the measure \(\mu_i^{+}\) of positive \(i\)-waves obtained by a comparison of
the solution to system (1.1) with a solution of Burgers’s equation with source terms.

**Theorem 5.1.** Consider system of balance laws (1.1) with initial data (1.2) and
assume that the system is strictly hyperbolic, each characteristic field is genuinely
nonlinear and the dissipativeness hypothesis (1.3) holds.

Let \(w = w(t,x)\) be the solution of the Cauchy problem for the scalar Burgers’s
equation with impulsive source term
\[
\partial_t w + \partial_x \left( \frac{w^2}{2} \right) + \beta w = -\kappa \text{sgn}(x) \cdot \frac{d}{dt} Q(u(t)),
\]
(5.1)
\[
w(0,x) = \text{sgn}(x) \cdot \sup_{|A| < 2|x|} \frac{\mu_i^{+}(A)}{2}
\]
(5.2)
with \(\beta, \kappa\) denoting some positive constants.

Then, for every solution \(u = u(t,x)\) of (1.1), (1.2) with small total variation, and
for every \(t \geq 0\), and \(i = 1, \ldots, n\),
\[
\mu_i^{+} \leq D_x w(t).
\]
(5.3)

This theorem together with the comparison lemma in Subsection 3.1 is critical
in reducing the proof of estimate (4.9) on rarefaction fronts to the case of the scalar
Burgers’s equation (5.1) with source. For related results on the positive waves for
systems of balance laws, we refer the reader to [19].

Now, the following lemma goes one step further; it provides the connection be-
tween a solution \(w\) to the scalar problem (5.1)–(5.2) with “a generalized rarefaction
wave” \(v\) solving
\[
v_t + v v_x + \beta v = 0
\]
(5.4)
centered at the origin as given in (5.6) below. In fact, to prove estimate (4.9), it suffices to study a much simpler case, the one having only one single centered “generalized rarefaction wave” to (5.4). This is the main idea in the proof of Proposition 5.3 in the end of the section.

**Lemma 5.2.** Let \( u \) be a solution of (1.1) defined for \( t \in [0, \tau] \) and \( \bar{w} \) as given in Theorem 5.1. Consider the quantity

\[
\bar{\sigma} = \frac{1}{2} \mu_0^+ (\mathbb{R}) - \kappa \int_0^\tau e^{\beta s} \frac{dQ}{ds}(u(s)) \, ds \tag{5.5}
\]

and the function

\[
v(t,x) = \begin{cases} 
  x e^{-\beta t} & \text{if } \frac{|x|}{\beta(1-e^{-\beta t})} \leq \bar{\sigma}, \\
  \text{sgn}(x) \bar{\sigma} e^{-\beta t} & \text{if } \frac{|x|}{\beta(1-e^{-\beta t})} > \bar{\sigma},
\end{cases} \tag{5.6}
\]

then

\[
\int_0^\tau \int_{-\infty}^\infty [w(t,x + \rho) - w(t,x - \rho)] w_x(t,x) \, dx \, dt \leq \int_0^\tau \int_{-\infty}^\infty e^{-\beta t} [v(t,x + \rho) - v(t,x - \rho)] v_x(t,x) \, dx \, dt. \tag{5.7}
\]

**Proof.** The proof follows ideas in [11], but additional work is required to treat the source term \( \beta \bar{w} \). The expression of \( Q \) employed in this proof is given by (3.2).

We first introduce a coordinate transformation \((t,x) \mapsto (t,\xi)\) as follows:

\[
x(t,\xi) = \frac{1}{\beta} \left(1 - e^{-\beta t}\right) e^{\beta t} \xi \tag{5.8}
\]

for \( \xi \in [0,\bar{\sigma}] \) and \( t \in [0,\tau] \). The transformation is motivated by the expression of the function \( v \) as given in (5.6).

Next, we define the functions

\[
\xi(t) \doteq \kappa \int_\tau^t e^{\beta s} \frac{dQ}{ds}(u(s)) \, ds, \quad \text{for } t \in [0,\tau] \tag{5.9}
\]

and

\[
t(\xi) \doteq \inf \left\{ t \geq 0 : \kappa \int_\tau^t e^{\beta s} \frac{dQ}{ds}(u(s)) \, ds \leq \xi \right\}, \quad \text{for } \xi \in [0,\bar{\sigma}] \tag{5.10}
\]

In account of (5.9)–(5.10), we consider the point \( y(t,\xi) > 0 \) given implicitly via the relation

\[
e^{\beta t} w(t,y(t,\xi)) = \xi - \xi(t) = \xi - \kappa \int_\tau^t e^{\beta s} \frac{dQ}{ds}(u(s)) \, ds \tag{5.11}
\]

for \( t \in [0,\tau] \) and \( \xi \in \left[\xi(t),\bar{\sigma}\right] \). It should be noted that relation (5.11) implies

\[
e^{\beta t} [w(t,\infty) - w(t,y(\xi,t))] = \bar{\sigma} - \xi \tag{5.12}
\]
Differentiating (5.11) with respect to $t$ and using (5.1), it follows

$$\frac{\partial y}{\partial t} = w(t, y(t, \xi)),$$

$$= e^{-\beta t} \left[ \xi - \kappa \int_{\tau}^{t} e^{\beta s} \frac{dQ}{ds} \, ds \right]. \tag{5.13}$$

Hence,

$$y(t, \xi) = \frac{1 - e^{-\beta t}}{\beta} \xi - \kappa \int_{\tau}^{t} \int_{\tau}^{s} e^{\beta s} \frac{dQ}{ds} \, ds \, ds + \alpha(\xi) \tag{5.14}$$

where $\alpha(\cdot)$ is an arbitrary function. Now, for $0 < \xi_1 < \xi_2 < \bar{\sigma}$ and $s > 0$, we compute

$$y(t(\xi_1) + s, \xi_2) - y(t(\xi_1) + s, \xi_1) = \frac{1 - e^{-\beta(t(\xi_1)+s)}}{\beta} (\xi_2 - \xi_1) + \alpha(\xi_2) - \alpha(\xi_1)$$

$$= y(t(\xi_1), \xi_2) - y(t(\xi_1), \xi_1) + e^{-\beta t(\xi_1)} \frac{1 - e^{-\beta s}}{\beta} (\xi_2 - \xi_1)$$

$$\geq e^{-\beta t(\xi_1)+s} [x(s, \xi_2) - x(s, \xi_1)]. \tag{5.15}$$

Using that both $w$ and $v$ are odd and nondecreasing functions and inequality (5.15), we employ the change of variables (5.11), (5.8) and expression (5.6) to estimate

$$\int_{0}^{\tau} \int_{-\infty}^{\infty} [w(t, x + \rho) - w(t, x - \rho)] v_\tau(t, x) \, dx \, dt =$$

$$= 2 \int_{0}^{\tau} \int_{\xi(t)}^{\sigma} [w(t, e^{\beta t} y(t, \xi) + \rho) - w(t, e^{\beta t} y(t, \xi) - \rho)] \, d\xi \, dt$$

$$\leq 4 \int_{0}^{\tau} \int_{\xi(t)}^{\sigma} [w(t, e^{\beta t} y(t, \xi) + \rho) - w^+(t, e^{\beta t} y(t, \xi) - \rho)] \, d\xi \, dt$$

$$= 4 \int_{0}^{\tau} \int_{e^{\beta t} |y(t, \xi)| - |y(t, \xi_2)| < \rho} e^{-\beta t} d\xi_1 \, d\xi_2 \, dt$$

$$\leq 4 \int_{0}^{\tau} \int_{|x(t, \xi_1)| - |x(t, \xi_2)| < \rho} e^{-\beta t} d\xi_1 \, d\xi_2 \, dt$$

$$= 4 \int_{0}^{\tau} \int_{\xi(t)}^{\sigma} e^{-\beta t} [v(t, x(t, \xi) + \rho) - v^+(t, x(t, \xi) - \rho)] \, d\xi \, dt$$

$$\leq 2 \int_{0}^{\tau} \int_{-\infty}^{\infty} e^{-\beta t} [v(t, x + \rho) - v(t, x - \rho)] v_\tau(t, x) \, dx \, dt. \tag{5.16}$$

The proof of bound (5.7) is complete. $\square$

**Proposition 5.3.** Consider a modified front tracking approximation $u(t, x)$ to system (1.1) with initial data (1.2) of small total variation and assume that the system is strictly hyperbolic and each characteristic field is genuine nonlinear. Then, for every $\tau > 0$, $u(t, \cdot)$ satisfies

$$\sum_{p, q \in \mathcal{P}, |p - q| \leq 8\sqrt{\tau}} \left| \sigma_p \sigma_q \right| dt = O(1)(\ln(2 + \frac{1 - e^{-\beta \tau}}{\beta}) + |\ln \epsilon|) \sqrt{\epsilon} \cdot TV \{\tilde{u}\}, \tag{5.17}$$
where $R_i$ is the set of rarefaction fronts of the $i$-th family and $\sigma_p$ denotes the strength of the front.

Proof. First, we consider the solution to (5.4) with a single centered rarefaction wave of strength $\sigma$ connecting the states $u^-$ and $u^+$. This solution is given by the expression

$$v(t, x) = \begin{cases} 
  e^{-\beta t} u^- & \text{if } \frac{x}{\beta(1 - e^{-\beta t})} < u^- \\
  e^{-\beta t} \omega(s) & \text{if } \frac{x}{\beta(1 - e^{-\beta t})} = \omega(s) \\
  e^{-\beta t} u^+ & \text{if } \frac{x}{\beta(1 - e^{-\beta t})} > u^+ .
\end{cases} \quad (5.18)$$

Let $\delta > 0$, then the oscillation satisfies

$$\text{Osc}\{v(t) : [y - \delta, y + \delta]\} \leq M \min\{\sigma, \frac{2\delta \beta}{1 - e^{-\beta t}}\} \quad (5.19)$$

and $\int |v_x(t, x)| \, dx \leq M \sigma e^{-\beta t}$, where $M$ is an upper bound of the length of the right eigenvectors $r_i(u)$. Set $t^* = -\frac{1}{\beta} \ln[1 - \frac{2\delta}{\sigma \beta}]$, then we compute the integral as follows

$$I(\tau) = \int_0^\tau \int \text{Osc}\{v(t) : [y - \delta, y + \delta]\} |v_x(t, y)| \, dy \, dt$$

$$\leq \int_0^{t^*} M^2 \sigma^2 e^{-\beta t} \, dt + \int_{t^*}^\tau M^2 \sigma 2\delta \frac{e^{-\beta t}}{\beta(1 - e^{-\beta t})} \, dt$$

$$= 2M^2 \sigma \delta \left(1 + \ln \frac{\sigma(1 - e^{-\beta \tau})}{2\beta \delta}\right). \quad (5.20)$$

Next, we consider the measures of positive waves $\mu^{i+}$ in the solution to (1.1). By Theorem 5.1 and Lemmas 3.3 and 5.2, we deduce that $I(\tau)$ controls an integral of the same form, but involving only the measures $\mu^{i+}$ of the solution $u$ to (1.1). Namely, if we set $\sigma = \text{Tot.Var.}\{\bar{u}\} < 1$, then for every $i = 1, \ldots, n$, the measures of positive waves $\mu^{i+}$ satisfy the estimate

$$\int_0^{\tau} (\mu^{i+} \otimes \mu^{i+})(\{(x, y) : |x - y| \leq \delta\}) \, dt = O(1) \delta \sigma \left[1 + \ln \frac{\sigma(1 - e^{-\beta \tau})}{2\beta \delta}\right]$$

$$= O(1) \left(\ln(2 + \frac{1 - e^{-\beta \tau}}{\beta}) + |\ln \delta|\right) \delta \text{TV}\{\bar{u}\}. \quad (5.21)$$

We remark here that the above estimate holds true for an exact solution to (1.1). We recall that, for the purpose of this manuscript, we denote by $u$ a modified front-tracking approximation. Hence, we claim that estimate (5.21) holds even in that case. Indeed, if $u$ is an exact solution and $u^{\varepsilon,s} \to u$ is a convergent subsequence of the modified front-tracking algorithm, then the corresponding measures of the $i$-waves converge weakly, but this is not true in general for the signed measures. Taking into consideration that on a fixed interval each sequence $u^{\varepsilon,s}$ may contain an alternating sequence of small positive and negative waves, that cancel only in the limit as $\varepsilon' \to 0, s \to 0$, (cf. Figure 2.4, p. 1089 in [11]) one can see that the solutions $u^{\varepsilon,s}$ need to be modified accordingly so that the weak convergence of the signed measures

$$\mu^{i+}_{\varepsilon,s,t} \to \mu^{i+}_t, \quad \text{as } \varepsilon' \to 0, s \to 0 \quad (5.22)$$
holds true for each \( t \) in a discrete set of times \( \left\{ \frac{\tau}{N}:j=0,1,\ldots,N \right\} \) with \( N \gg \varepsilon^{-1} \). Consequently, we obtain an arbitrarily accurate front tracking approximation satisfying an estimate analogous to (5.21), namely

\[
\int_0^\tau (\mu^+_i \otimes \mu^+_i)(\{(x,y):\vert x-y \vert \leq 8\sqrt{\varepsilon}\})dt = O(1) \left( \ln(2 + \frac{1 - e^{-\beta \tau}}{\beta}) + \ln \varepsilon \right) \sqrt{\varepsilon} TV\{\tilde{u}\},
\]

where the \( \delta \) appearing in (5.21) has been replaced by \( 8\sqrt{\varepsilon} \) in anticipation of the proof of Property 4. Without loss of generality, these approximations are denoted again by \( u \). Thus, by convergence, estimate (5.17) follows. We refer the reader to Bressan-Yang [10] for further details.

6. Estimates on Shock Fronts. The main objective in this section is the establishment of estimate (4.10).

**Lemma 6.1.** The integral in (2.7) relative to shock fronts \( S(t) \) at time \( t \) is of the order

\[
\sum_{p \in S(t)} \int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} |v_l + A(v)v_x + g(v) - \varepsilon v_{xx}|dx = O(1) \cdot |\varepsilon(1 + |\ln \varepsilon|) + \sqrt{\varepsilon} e^{-\beta t} TV\{\tilde{u}\} + O(1) \left( \sum_{x_{q1},x_{q2} \in O_p(t), \ |x_{q1} - x_{q2}| \leq 2\sqrt{\varepsilon}} |\sigma_{q1}\sigma_{q2}| - \sum_{x_r \in O_p(t), r \in BS} |\sigma_r|^2 \right). \tag{6.1}
\]

**Proof.** To begin with, we estimate the running error in (6.1) related to big shocks, namely the integral

\[
E_{BS}(t) := \sum_{p \in BS(t)} \int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} |v_l + A(v)v_x + g(v) - \varepsilon v_{xx}|dx. \tag{6.2}
\]

For fixed \( p \in BS \), we assume that there is only one wave front (the big shock \( p \)) within the interval

\[
O_p(t) := [x_p(t) - 2\sqrt{\varepsilon}, x_p(t) + 2\sqrt{\varepsilon}]. \tag{6.3}
\]

By definition of the viscous shock profile \( \omega_p^\varepsilon(\zeta) \), the error relative to the shock at \( x_p \) can be written as

\[
E_p(t) = \left\{ \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \left[ A(\omega_p^\varepsilon(\theta(\zeta))) - \lambda_p \frac{\partial}{\partial s} \omega_p^\varepsilon(\theta(\zeta)) \theta'(\zeta) \right]
- \varepsilon \frac{\partial}{\partial s} \omega_p^\varepsilon(\theta(\zeta)) \theta''(\zeta)
- \varepsilon \frac{\partial^2}{\partial s^2} \omega_p^\varepsilon(\theta(\zeta)) (\theta'(\zeta))^2 + g(\omega_p^\varepsilon(\theta(\zeta))) \right\} d\zeta. \tag{6.4}
\]

Using the bounds on the derivatives

\[
\left| \frac{\partial}{\partial r} \omega_p^\varepsilon(r) \right| = O(1) \cdot \left| \frac{\sigma_p}{\varepsilon} \right| e^{-\frac{|r\sigma_p|}{\varepsilon}}, \quad \left| \frac{\partial^2}{\partial r^2} \omega_p^\varepsilon(r) \right| = O(1) \cdot \left| \frac{\sigma_p^3}{\varepsilon^2} \right| e^{-\frac{|r\sigma_p|}{\varepsilon}},
\]

for further details.
and (2.18), integral (6.4) can be estimated as follows
\[
E_p(t) = O(1) \cdot \int_{\frac{\sqrt{\varepsilon}}{2}}^{\frac{\sqrt{\varepsilon}}{2}} \exp \left\{ -\frac{\|p\|}{\varepsilon} \theta(\zeta) \right\} \cdot \left( \frac{\|p\|^2}{\varepsilon} \theta'(\zeta) + \frac{\|p\|^3}{\varepsilon} \theta''(\zeta) \right) + g(\omega_p^\varepsilon(\theta(\zeta))) d\zeta \\
= O(1) \cdot \left[ \int_{\frac{\sqrt{\varepsilon}}{2}}^{\frac{\sqrt{\varepsilon}}{2}} \exp \left\{ -\frac{\|p\|}{4(\sqrt{\varepsilon} - \zeta)} \right\} \frac{\|p\|^3}{(\sqrt{\varepsilon} - \zeta)^3} d\zeta \right] \\
= O(1) \cdot \int_{\frac{\sqrt{\varepsilon}}{2}}^{\frac{\sqrt{\varepsilon}}{2}} \exp \left\{ -\frac{\|p\|s}{4} \right\} \|p\|^3 s^3 \frac{ds}{s^2} + O(1) \int_{\frac{\sqrt{\varepsilon}}{2}}^{\frac{\sqrt{\varepsilon}}{2}} \|p\| \exp \left\{ -\frac{\|p\|}{\varepsilon} \right\} d\zeta \\
= O(1) \cdot \|p\| \exp \left\{ -\frac{\|p\|}{2\varepsilon} \right\} \left( 1 + \frac{2\|p\|}{\varepsilon} \right) + O(1) \cdot \sqrt{\varepsilon}\|p\|. 
\tag{6.5}
\]
Since \( p \in BS \), we have
\[
\|p\| \geq \frac{\varepsilon}{2} = 2\sqrt{\varepsilon}\|\varepsilon\|, 
\]
and estimate (6.5) implies
\[
E_p(t) = O(1) \cdot |\varepsilon(1 + |\varepsilon|) + \sqrt{\varepsilon}|\|p\|. 
\tag{6.6}
\]
It turns out that the error estimate (6.6) should take into account the presence of the other wave fronts within the intervals \( O_p(t) \). Indeed, for every point \( x_p \) where the big shock is located, we have
\[
E_p(t) := \int_{x_p - \sqrt{\varepsilon}}^{x_p + \sqrt{\varepsilon}} \left| v_t + A(v)v_x + g(v) - \varepsilon v_{xx} \right| dx = O(1) \cdot |\varepsilon(1 + \ln \varepsilon) + \sqrt{\varepsilon}|\|p\| \\
+ O(1) \left( \sum_{x_{q_1}, x_{q_2} \in O_p(t), \|x_{q_1} - x_{q_2}\| \leq 2\sqrt{\varepsilon}} |\|p\| - \sum_{x_r \in O_p(t), r \in BS} |\|p\|^2| \right). 
\tag{6.7}
\]
and the proof is complete.

Following the approach introduced in [11], we employ the functionals, \( Q^l \), \( Q^m \) and \( Q^r \), which account for products \( |\|p\|| |\|q\| \) (a) of fronts of different families, (b) where \( \|p\| \) is a big shock and \( \|q\| \) is a rarefaction of the same family, and (c) of shocks of the same family, respectively. Then, it can be shown that the functional \( \hat{Q} \) defined in (3.5) is of the order
\[
\hat{Q}(u(t)) = O(1) \sqrt{\varepsilon}\ln \varepsilon|TV\{u(t)\} \\
= O(1) \sqrt{\varepsilon}\ln \varepsilon e^{-\beta t} \delta_0. 
\]
Moreover, we have the following result on the increase of \( \hat{Q} \):

**Lemma 6.2.** There are positive constants \( C_1 \), \( C_2 \) and \( C_3 \) such that if the total variation of \( u(t) \) is sufficiently small, then at each interaction time \( t \), the functional \( \hat{Q} \) defined in (3.5) satisfies the following: if a new big shock of strength \( |\|p\| > 2\sqrt{\varepsilon} |\varepsilon| \) is created, then
\[
\Delta \hat{Q} := \hat{Q}(t+) - \hat{Q}(t-) = O(1) \cdot \sqrt{\varepsilon}\ln \varepsilon|\|p\|, 
\tag{6.8}
\]
otherwise,
\[
\Delta \hat{Q} \leq 0. 
\tag{6.9}
\]
Proof. By the modified front tracking algorithm, within each time strip \((t^*_k, t^*_k+1)\), the interaction of waves at time \(t\) is given by the homogeneous system, which immediately implies that one can follow the steps in Bressan et al. [11] to prove the result.

Combining estimates (6.8), (6.9) with (6.7), and following similar line of argument as in [11], we arrive at estimate (4.10). It should be noted that the factor \((1 - e^{-\beta \tau})/\beta\) is due to the exponential decay in time of the total variation of \(u\).

Appendix. The Modified Front Tracking Algorithm. In this appendix, we state existing results in the theory of hyperbolic systems of balance laws and provide a description of the modified front-tracking algorithm.

A.1. The Semigroup. Given a system of hyperbolic conservation laws, \(g = 0\), with initial data of small total variation, existence and stability of solutions have been established in a series of articles. We refer the reader to the articles [5, 15, 8, 9, 4] and the references therein. For the purpose of this manuscript, we are interested in solutions of (1.1) obtained by building a family of Lipschitz semigroups \(L_t\) as the \(s \to 0\) limit of an approximate sequence of operators \(L^*_t\) constructed by a suitably modified wave front tracking algorithm.

Roughly speaking, if we denote by \(\mathcal{D}\) the domain

\[
\mathcal{D}(\delta) = \text{cl}\{u \in L^1 \cap BV(\mathbb{R}; \mathbb{R}^n) : u \text{ is piecewise constant and } \Upsilon(u) < \delta\},
\]

where the closure is taken in the \(L^1\)-topology, then there exist \(\delta_S, L_S > 0\) and a unique dynamical system (semigroup)

\[
S : [0, +\infty) \times \mathcal{D}(\delta_S) \to \mathcal{D}(\delta_S)
\]

satisfying a series of properties [5, 15, 8], such that if \(u_0 \in \mathcal{D}\) is piecewise constant, then, for \(t\) sufficiently small, \(u(\cdot, t) = S_t u_0\) coincides with the solution obtained by piecing together the standard entropy solutions of the Riemann problems determined by the jumps of \(u_0\) associated with the homogeneous system

\[
\partial_t u + \partial_x f(u) = 0
\]

Moreover, for each \(\delta \leq \delta_S\) the domains \(\mathcal{D}(\delta)\) are invariant under the evolution of the semigroup \(S\).

A Lipschitz evolution operator \(\mathcal{L}\) for the hyperbolic balance laws (1.1) has been constructed in [2]. Under the diagonally dominance assumption, the evolution operator \(\mathcal{L} : [0, +\infty) \times \mathcal{D} \to \mathcal{D}\) turns out to be a semigroup defined on a suitable domain \(\mathcal{D}\). Moreover, the evolution operator \(\mathcal{L}\) satisfies the following key estimate:

\[
\|\mathcal{L}_h u_0 - S_h u_0 + h \cdot g(u_0)\|_{L^1} = O(1)h^2 \text{ as } h \to 0.
\]

We state below the theorem which determines the global existence of solutions for Cauchy problem for (1.1)–(1.2) as established in [2].

**Theorem A.1.** Under the dissipativeness assumption (1.3) with \(g(u^*) = 0\), there exist a closed domain \(\mathcal{D} \subset L^1(\mathbb{R}, \mathbb{R}^n) \cap BV(\mathbb{R}, \mathbb{R}^n)\), containing all \(L^1\)-functions with sufficiently small total variation, positive constants \(\beta, C_0, L_P > 0\) and a unique semigroup \(\mathcal{L} : [0, +\infty) \times \mathcal{D} \to \mathcal{D}\) satisfying for each \(u_0, v_0 \in \mathcal{D}\) and all \(s, t \geq 0\) the following properties:
On the other hand, assume that a Lipschitz map \( u \) that estimates (A.1)–(A.2) become

\[
\begin{align*}
0 = & \text{\emph{mesh time interval } } \Delta \\
\text{the homogeneous system } & (A.5) \text{ and } \\
\text{\emph{approximately the ordinary differential equation}} \\
\text{initial data } & \bar{\omega}.
\end{align*}
\]

Then a modified front-tracking method. We refer the reader to [15, 2] for further details.

\text{
In the following construction, the source term is handled by the \emph{operator splitting} method.
}

\text{
For a fixed parameter } \varepsilon' > 0 \text{ and a time step } s = \Delta t \equiv t_k^* - t_{k-1}^*, k = 1, \ldots, m, \text{ we start at time } t_0 = 0, \text{ by approximating the initial data by a piecewise constant data } \bar{u}^s. \text{ For convenience, we drop the index } s \text{ and denote this piecewise constant initial data by } \bar{u}. \text{ Roughly speaking, we pass from } t = t_k^* \text{ to } t = t_{k+1}^*, \text{ by first solving approximately the ordinary differential equation}
\]

\[
\frac{\partial}{\partial t} u + g(u) = 0 \quad \text{on} \quad (t_k^*, t_{k+1}^*)
\]

and then solving independently, the conservation law

\[
\frac{\partial}{\partial t} u + \partial_x f(u) = 0. \tag{A.5}
\]

More precisely, the process is described as follows. Let \( S \) be the semigroup for the homogeneous system (A.5) and \( g \) denote a fixed parameter \( g \in (0, \varepsilon') \). Given a mesh time interval \( \Delta t = s > 0 \) and a function \( \bar{u} \in D(g) \), the approximate operator \( L^s \) is defined as follows: At } t = 0,

\[
L^s_0 \bar{u} = \bar{u}.
\]
For every $t \in (ks,(k+1)s)$, and for every $k \geq 0$, we solve the homogeneous problem (A.5) by applying the semigroup operator $S_{t-ks}$ to the solution $L_{ks}^t$, namely
\[ L_t^s \bar{u} = S_{t-ks} L_{ks}^s \bar{u}. \]

To estimate the solution at time $t_k^* = ks$, $k \geq 1$, we add to the value of the solution $L_{ks}^t u$ estimated as
\[ L_{ks}^t u = \lim_{t \to ks^-} L_t^s \bar{u} = S_{s} L_{ks}^s \bar{u}, \]
the value of an approximate solution to the differential equation $\partial_t u + g(u) = 0$ evaluated at time $t = ks-$, namely
\[ L_{ks}^t \bar{u} = \lim_{t \to ks^-} L_t^s \bar{u} - s \cdot g(L_{ks}^s \bar{u}). \]

Using the above operating splitting method, given a mesh size $s = \Delta t$, an approximating sequence $L_t^s$ is constructed for all times $t > 0$ and it is shown that as $s \to 0$, $L_t^s$ converges to the Lipschitz semigroup $L$ associated with the Cauchy problem for (1.1). Under the dissipativeness condition (1.3), the Lipschitz evolutionary operator $L^s$ is an approximated semigroup defined for all $t \geq 0$ that satisfies
\[ L_t^s \bar{u} \in \mathcal{D}(\min(1,e^{2s\rho}e^{-2\beta t}) \cdot g), \quad \Upsilon(L_t^s \bar{u}) \leq e^{-\beta t} \Upsilon(\bar{u}) \]
\[ \text{TV} L_t^s \bar{u} \leq C \min(1,e^{2s\rho}e^{-2\beta t}) \text{TV} \{\bar{u}\}, \]
for suitable positive constants $C$ and $\beta$ independent of $s$. It should be noted that $0 < \beta < \bar{\beta}$ and $\bar{\beta} - \beta = O(\text{TV} \bar{u})$. Here, the functional $\Upsilon$ is expressed using the continuous Glimm functionals (3.1)-(3.2).

Similarly, for a general source $g$, for any $\bar{u}$ within the domain $\mathcal{D}(g)$, the approximate trajectories $L_t^s \bar{u}$ are defined within the interval $[0,T]$ and it holds
\[ L_t^s \bar{u} \in \mathcal{D}(e^{\gamma t} g), \quad \Upsilon(L_t^s \bar{u}) \leq e^{\gamma t} \Upsilon(\bar{u}) \]
\[ \text{TV} L_t^s u_0 \leq C_1 e^{\gamma t} \text{TV} \{\bar{u}\}, \]
for suitable positive constants $C_1$ and $\gamma$.

The above construction generates a sequence of approximate solutions $u^s(\cdot, t) = L_t^s \bar{u}$, $s = \Delta t$ such that as $s \to 0$, we can pass into the limit and the sequence $u^s$ converges to the solution of the Cauchy problem from (1.1) [2]. Now, the solution $u^{\varepsilon,s}(x,t)$ obtained by the modified front tracking algorithm can be obtained by approximating the semigroup $S$ by the semigroup $S^{\varepsilon}$ associated with the front-tracking approximate solutions for the system of conservation laws ($g = 0$). Here, $\varepsilon$ denotes the front-tracking parameter and as $\varepsilon \to 0$, $u^{\varepsilon}(\cdot, t) = S_t^{\varepsilon} w_0$ converges to the entropy weak solution of the system of conservation laws ($g = 0$). For more details on the front-tracking method, we refer the reader to Bressan [5], Dafermos [15] and Holden-Risebro [22]. In other words, at every time step $(t_k^*, t_{k+1}^*) = (ks, (k+1)s)$, we obtain a piecewise constant function $u^{\varepsilon,s}(t) = L_t^{\varepsilon,s} \bar{u}$ using the front-tracking approximate semigroups $S^{\varepsilon}$, and at times $t_k^* = ks$, we apply the operating splitting method as described above on the modified operator $L^{\varepsilon,s}$ at $t = ks-$. In this way,
\textit{u^{\varepsilon,s}(t) = \mathcal{L}_t^{\varepsilon,s} u^0} is a globally defined piecewise constant function with a finite number of discontinuities that enjoys all properties of a front tracking approximation and as \(\varepsilon' \to 0\), \(u^{\varepsilon',s}(t)\) converges to \(u^s(t) = \mathcal{L}_t^s \bar{u}\). Thus, as \(\varepsilon' \to 0^+\) and \(s \to 0\), we can extract a subsequence such that \(u^{\varepsilon',s} \to u\), where \(u\) is an entropy weak solution to the Cauchy problem for the balance law (1.1).

It should be noted that the above construction is global in time for dissipative source \(g\) and it can be extended up to \(t = T > 0\) for general source \(g\). Also, one could construct a sequence of approximate solutions by employing the random choice method of Glimm [18] (see Dafermos and Hsiao [17]) or some other approximation scheme to approximate the semigroup \(\mathcal{S}\) within each time interval.

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