MATH/AMSC 673 (Fall 2004)
PARTIAL DIFFERENTIAL EQUATIONS
TAKE HOME FINAL

Problem 1. In order to solve the semilinear heat equation

$$\begin{cases} u_t - \Delta u = f(u) & \mathbf{R}^n \times (0, \infty) \\ u(x,0) = g(x) & x \in \mathbf{R}^n, \end{cases} \tag{1}$$

we may resort to the Duhamel's principle and write the nonlinear integral equation

$$u(x,t) = \int_{\mathbf{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbf{R}^n} \Phi(x-y,t-s) f(u(y,s))\, dyds. \tag{2}$$

(a) Explain the relation between (1) and (2).
(b) Show that the integral equation has a unique solution for all $T > 0$.
(c) Use an energy argument to show that (1) has only one solution.

Problem 2. Let $h \in C(\mathbf{R}^3)$ and $u(x,t)$ be a solution of the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0, \\ u(x,0) = 0, \quad u_t(x,0) = h(x). \end{cases}$$

Show that

$$\int_0^\infty u^2(0,t)\, dt \le \frac{1}{4\pi} \int_{\mathbf{R}^3} h^2(x)\, dx.$$

Problem 3. Let $g : \mathbf{R} \to \mathbf{R}$ be bounded and C^1 function such that

$$1 < G = \max_{x \in \mathbf{R}} g'(x) < \infty.$$

Consider the initial value problem

$$\begin{aligned} &u_t - u u_x + u = 0, \quad \text{in } \mathbf{R} \times (0,T) \\ &u(x,0) = g(x) \quad \text{for } x \in \mathbf{R}. \end{aligned}$$

(a) Construct a classical solution for all $x \in \mathbf{R}$ and small $t > 0$.
(b) Show that there is no C^1 solution defined for all $t > 0$. Find the time T (in terms of G) at which the classical solution breaks down.

Problem 4. Consider the Cauchy problem for Burgers' equation,

$$u_t + u u_x = 0$$

with the initial data

$$u(x,0) = \begin{cases} 1, & \text{for } |x| > 1, \\ |x| & \text{for } |x| \le 1. \end{cases}$$

(a) Sketch the characteristics in the (x, t) plane. Find a classical solution (continuous and piecewise C^1) and determine the time of breakdown (shock formation).
(b) Find a weak solution for all $t > 0$ containing a shock curve. Note that the shock does not move with constant speed. Therefore, find first the solution away from the shock. Then use the Rankine-Hugoniot condition to find a differential equation for the position of the shock given by $(x = s(t), t)$ in the (x, t) plane.

Problem 5. Verify that the viscous approximation of Burgers' equation

$$u_t + uu_x = \epsilon u_{xx}$$

has a traveling wave solution of the form $u(x, t) = v(x - st)$ with

$$v(y) = u_R + \frac{1}{2}(u_L - u_R)\left(1 - tanh\left(\frac{(u_L - u_R)y}{4\epsilon}\right)\right),$$

where $s = \frac{u_L + u_R}{2}$ is the shock speed given by the Rankine Hugoniot condition. Sketch this solution. What happens as $\epsilon \to 0$?

Problem 6. If $0 \le u \le 1$ represents the (normalized) density of cars in a street, a simple model for traffic flow is governed by the conservation law:

$$u_t + f(u)_x = 0, \quad f(u) = u(1 - u).$$

Therefore the speed of cars $v(u) = 1 - u$ decreases as their density u increases (note that f is concave); $u = 0$ stands for no cars (empty road), and $u = 1$ for bumper-to-bumper cars (full road). Consider the initial condition

$$u(x,0) = \begin{cases} 0 & x > 1 \\ 1 & 0 < x < 1 \\ a & x < 0. \end{cases}$$

This corresponds to a traffic stream that has been partly stopped at a traffic light, and starts again at $t = 0$, whereas $0 < a < 1$ is the density of a uniform stream of cars approaching from the left.
(a) Find the entropy weak solution for $0 < t < 1/(1 - a)$, which is a combination of shocks and rarefaction waves. Sketch the characteristics and the shock discontinuity in (x, t) - plane, and the solution $u(x, t)$ as a function of $x \in \mathbf{R}$. Verify the Rankine Hugoniot condition across any discontinuities.
(b) Construct the entropy weak solution for $t \ge 1/(1 - a)$. To this end, find an ODE for the shock discontinuity and solve it. Sketch the characteristics and the shock discontinuity in (x, t)-plane and the solutions $u(x, t)$ as a function of $x \in \mathbf{R}$.