Plan

- Integrable and nearly integrable systems, FPU paradox, KAM theorem, Nekhoroshev stability.
- **Warm up**: KAM theorem in low dimension, Newton’s method, Moser-Levi’s proof of KAM.
- KAM theorem: Herman-Fejoz’s proof via an implicit function theorem.
- Application: stability of planetary motions, the semi-classical asymptotics for Schrödinger operators.
Let \( H_0(I) \) – integrable and \( H_\varepsilon(\theta, I) = H_0(I) + \varepsilon H_1(\theta, I) \) – a perturbation.

**KAM Theorem**

Let \( H_0(I) \) be real analytic and nondegenerate, i.e. Hessian \( \det \partial^2 H_0 \neq 0 \). Let \( H_\varepsilon(\theta, I) \) be a real analytic perturbation. Then most initial conditions in an open bounded set \( \mathbb{T}^n \times B \subset \mathbb{T}^n \times \mathbb{R}^n \) have quasiperiodic orbits filling an analytic \( n \)-dimensional torus. In particular, most solutions are bounded.
Let $\tau, \gamma > 0$. $\omega \in \mathbb{R}^n$ is called $(\tau, \gamma)$-diophantine if for all $k \in \mathbb{Z}^n \setminus 0$ we have $|\omega \cdot k| \geq \gamma |k|^{-(n-1)-\tau}$. Denote this set $\mathcal{D}_{\tau,\gamma}$.

**KAM Theorem**

Let $H_0(I)$ be real analytic and nondegenerate, i.e. Hessian $\det \partial^2 H_0 \neq 0$. Let $H_\varepsilon(\theta, I)$ be an analytic perturbation and $\varepsilon$ small.

- (one torus) for each $\omega \in \mathcal{D}_{\tau,\gamma}$ the system $H_\varepsilon$ has an analytic invariant $n$-dim’l torus $T^n_\omega$ with the flow on $T^n_\omega$ conjugate to $\dot{\theta} = \omega$.

- (positive measure of tori) the union of tori has measure $1 - O(\sqrt{\varepsilon})$ of the total measure.

- (global normal form) there is an analytic symplectic map $\Phi_\varepsilon$ s. t. $K_\varepsilon(\theta, I) = H_\varepsilon \circ \Phi_\varepsilon(\theta, I) = K_0(I) + \exp(-C\varepsilon^{-1}/\varrho)R(\theta, I)$, where $\varrho > 1$ and $R$ vanishes on $\Phi_\varepsilon^{-1}(\bigcup_{\omega \in \mathcal{D}_{\tau,\gamma}} T^n_\omega)$.

We focus on $n = 2$ and “one torus” part!
Plan of Lecture 2

- One torus KAM theorem.

- *Reduction principle:* from a $n$ degree of freedom nearly integrable system to a $2(n - 1)$-dim’l exact symplectic map.

- Appearance of exact twist maps for $n = 2$.

- Find an **Euler-Lagrange funct’l eqn** $E_\omega(u, H) = 0$ on $u$ such that

  $$\exists \text{ a solution } u_\varepsilon \text{ for } E_\omega(u_\varepsilon, H_\varepsilon) = 0 \text{ near a solution } E_\omega(u_0, H_0) = 0$$

  $$\iff \exists \text{ an invariant torus } \mathcal{T}_\omega \text{ for } H_\varepsilon.$$ 

- Using **Newton’s method** solve $E(u_\varepsilon, H_\varepsilon) = 0$ knowing that a solution $u_0$ to $E(u_0, H_0) = 0$. 

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Plan of Lecture 2

- One torus KAM theorem.

- Reduction principle: from a $n$ degree of freedom nearly integrable system to a $2(n - 1)$-dim’l exact symplectic map.

- Appearance of exact twist maps for $n = 2$.

- Find an **Euler-Lagrange functional eqn** $E_\omega(u, H) = 0$ on $u$ such that
  
  $\exists$ a solution $u_\varepsilon$ for $E_\omega(u_\varepsilon, H_\varepsilon) = 0$ near a solution $E_\omega(u_0, H_0) = 0$  
  $\iff \exists$ an invariant torus $T_\omega$ for $H_\varepsilon$.

- Using **Newton’s method** solve $E(u_\varepsilon, H_\varepsilon) = 0$ knowing that a solution $u_0$ to $E(u_0, H_0) = 0$. 
Reduction principle

Consider \( H_\varepsilon(\theta, I) = H_0(I) + \varepsilon H_1(\theta, I) \)
\( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n, \ I = (I_1, \ldots, I_n) \) and equations of motion

\[
\begin{cases}
\dot{\theta} = \partial_I H_0(I) + \varepsilon \partial_I H_1(\theta, I) \\
\dot{I} = -\varepsilon \partial_\theta H_1(\theta, I)
\end{cases}
\]

Suppose \( \partial_{I_n} H_0 \neq 0 \) for all \( I \in U \). Let \( E \) be value of energy. Using an implicit function theorem determine \( I_n(\theta, I_1, \ldots, I_{n-1}, E) \) so that

\[
H_\varepsilon(\theta, I_1, \ldots, I_{n-1}, I_n(\theta, I_1, \ldots, I_{n-1}, E)) = E
\]

Since \( \dot{\theta}_n \neq 0 \), for all \( (\theta, I) \in \mathbb{T}^n \times U \) the following (Poincare return) map is well-defined

\[
F_\varepsilon : (\theta_1, \ldots, \theta_{n-1}, I_1, \ldots, I_{n-1}) \rightarrow (\theta'_1, \ldots, \theta'_{n-1}, I'_1, \ldots, I'_{n-1}).
\]

It turns out \( F_\varepsilon \) is an exact symplectic map.
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It turns out $F_\varepsilon$ is an exact symplectic map.
Twist maps for $n = 2$ and convex Hamiltonians

Denote $\Lambda = \mathbb{T} \times \mathbb{R} \ni (\theta, I)$. Call $F : \Lambda \to \Lambda$ an (exact area-preserving) **twist map** if

- (area-presentation) $F$ preserves the area $d\theta \wedge dl$.
- (exact) for any non-contractible curve $\gamma \subset \Lambda$ area under $\gamma = \text{area under } F(\gamma)$.
- (twist) for any vertical curve $\Gamma_a = \{(\theta, I) : \theta = a\} \subset \Lambda$ the image $F((a, I))$ is strictly monotone in $l$.

**Theorem (Moser)**

Any twist map $F : (\theta, I) \to (\theta', I')$ of the annulus can be given by a time-periodic (non-autonomous) Hamiltonian flow, induced by a convex in action Hamiltonian $H(\theta, I, t)$, i.e. $\partial_{II}^2 H(\theta, I, t) > 0$. Namely, $F$ coincides with the time-one-map of the Hamiltonian flow associated to $H$. 
Generating functions and twist maps

A smooth function $h(\theta, \theta')$, $\theta, \theta' \in \mathbb{R}$ is called generating if

- (twist) $\partial_{\theta\theta}^2 h(\theta, \theta') < 0$ for all $\theta, \theta' \in \mathbb{R}$;
- (periodicity) $h(\theta + 1, \theta' + 1) = h(\theta, \theta')$ for all $\theta, \theta' \in \mathbb{R}$.

A generating funct. $h$ defines a twist map $F : (\theta, I) \rightarrow (\theta', I')$ as follows:

$$\begin{cases} 
\partial_{\theta} h(\theta, \theta') = -I \\
\partial_{\theta'} h(\theta, \theta') = I'
\end{cases} \quad (1)$$

**Theorem**

Any smooth (exact area-preserving) twist map $F$ possesses a generating function $h$ such that the map $F$ is given by (1).

For the standard map we have $h_{\varepsilon}(\theta, \theta') = \frac{(\theta' - \theta)^2}{2} + \frac{\varepsilon}{2\pi} \cos 2\pi \theta$.

$$\begin{cases} 
\theta - \theta' + \varepsilon \sin 2\pi \theta = -I \\
\theta' - \theta = I'
\end{cases} \quad \begin{cases} 
\theta' = \theta + I + \varepsilon \sin 2\pi \theta \\
I' = I + \varepsilon \sin 2\pi \theta.
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An Euler-Lagrange equation for an invariant curve

Let $\gamma = \{ \theta = u(\phi), \ l = v(\phi) : \phi \in \mathbb{R} \}$ a closed parametric curve in the covering $(\theta, l)$-plane of $\Lambda$ with $u(\phi) - \phi$ and $v(\phi)$ of period 1.

**The invariance condition**: $F(u(\phi), v(\phi)) = (u(\phi + \omega), v(\phi + \omega))$ (2)

with a strictly monotone $u$ in $\phi$. $F|_{\gamma}$ is conjugate to a rotation by $\omega$.

**Theorem**

The curve $(u(\phi), v(\phi))$ satisfies the invariance condition (2) with $F$ given by (1) if and only if the horizontal function $u(\theta)$ satisfies the second order difference equation

$$E_\omega(u(\phi)) = \partial_1 h(u(\phi), u(\phi + \omega)) + \partial_2 h(u(\phi - \omega), u(\phi)) = 0.$$ 

This equation is the Euler-Lagrange equation for the variational problem $\delta \int_0^1 h(u(\phi), u(\phi + \omega))d\phi = 0$ (Percival’s variational principle).
An Euler-Lagrange equation for an invariant curve

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**Theorem**

Assume that $E_\omega(u_0)$ is small, then there is a unique solution $u$ near $u_0$ of $E_\omega(u) = 0$ with an analytic function $u(\phi) - \phi$ of zero mean value.

In particular, the twist map $F$ corresponding to the generating function $h$ has an invariant curve $\gamma$, given by

$$\gamma = \{(u(\phi), v(\phi) = \partial_1 h(u(\phi), u(\phi + \omega))): \phi \in \mathbb{R}\}.$$
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Denote by $W_r$ the set of 1-periodic real analytic functions of $\phi$ bounded on the strip $|\text{Im} \phi| \leq r$. Introduce the maximum norm

$$\|f\|_r := \sup_{|\text{Im} \phi| \leq r} |f(\phi)|.$$  

Expand $f$ in Fourier series: $f(\phi) = \sum_{k \in \mathbb{Z}} f_k \exp(ik\phi)$.

$$f(\text{Re} \phi + i\text{Im} \phi) = \sum_{k \in \mathbb{Z}} f_k \exp(ik\text{Re} \phi) \exp(-k\text{Re} \phi).$$

If $f \in W_r$, then $|f_k| \leq \|f\|_r \exp(-|k|r)$ for all $k \in \mathbb{Z}$.

As decay of Fourier coeffs deteriorates strip of analyticity is shrinking.

For $\omega \in \mathbb{R}$ denote $u^\pm(\phi) = u(\phi \pm \omega)$ and by

$$\nabla_\omega u(\phi) = u^+(\phi) - u(\phi), \quad \nabla^*_\omega u(\phi) = u(\phi) - u^-(\phi), \quad \Delta_\omega u(\phi) = \nabla^*_\omega \nabla_\omega u(\phi).$$
Periodic real analytic functions

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\]
KAM theorem for twist maps

**KAM Theorem**

Let $h$ be real analytic in a proper domain $D$ and $\min |\partial_{12} h| \geq \kappa$ for some $\kappa > 0$. Let $\omega \in \mathbb{R}$ be $(\tau, \gamma)$-diophantine, i.e. $|\omega - p/q| \geq \gamma |q|^{-2-\tau}$ for $\forall p, q \neq 0$. Let $u_0 \in W_r$ be an approx. solution of zero mean value and $\|E_\omega(u_0)\|_r \leq \delta$. Then for $\delta$ small enough there is a unique solution $E_\omega(u) = 0$ with $u \in W_{r/2}$ near $u_0$ and $u(\phi) - \phi$ has zero mean value.

- The twist map given by $h$ does **not** need to be nearly integrable.
- For the standard map $h_\varepsilon(\theta, \theta') = \frac{(\theta' - \theta)^2}{2} + \frac{\varepsilon}{2\pi} \cos 2\pi \theta$, $\partial_{12} h \approx -1 \Rightarrow$ this Theorem applies.
- For the standard map the Euler-Lagrange equation is $E_\omega(u) = u^+ - 2u + u^- + \varepsilon \sin 2\pi u(\phi) = 0$ (**highly nonlinear**).
Newton’s method

Goal: Suppose $E_\omega(u_0) = E_0$ is small. Using Newton’s method find $E_\omega(u) = 0$ for some $u$ near $u_0$.

Newton Approximation procedure: Expand

$$E(u + v) = E(u) + DE(u)v + Q(v), \; Q(v) \sim |v|^2 \text{ – quadr. error.}$$

Solve so-called homological equation

$$DE(u)v = -E(u).$$

Step 1. Solve $DE(u_0)v_0 = -E(u_0)$. We have for $u_1 = u_0 + v_0$

$$E(u_1) = E(u_0) + DE(u_0)v_0 + Q(v_0) = Q(v_0) \sim E_0^2.$$

Step n. Solve $DE(u_n)v_n = -E(u_n)$. We have for $u_{n+1} = u_n + v_n$

$$E(u_{n+1}) = E(u_n) + DE(u_n)v_n + Q(v_n) = Q(v_n) \sim E^2(u_n) \sim E_0^{2^n}.$$

The solution $u = u_0 + \sum_{n=1}^{\infty} v_n$ is well-defined as $v_n \to 0$ superexponentially fast in $n$. 
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Newton’s method

**Goal:** Suppose $E_\omega(u_0) = E_0$ is small. Using Newton’s method find $E_\omega(u) = 0$ for some $u$ near $u_0$.

Newton Approximation procedure: Expand

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The solution $u = u_0 + \sum_{n=1}^{\infty} v_n$ is well-defined as $v_n \to 0$ superexponentially fast in $n$. 
Solving cohomological equation

In order to solve the Euler-Lagrange functional equation

\[ E_\omega(u) = 0 \quad \text{for some } u \text{ near } u_0. \]

We plan to solve the cohomological equation

\[ DE(u)v = (\partial_{11} h + \partial_{22} h_-)v + \partial_{12} h^+ v^+ + \partial_{12} h^- v^- = -E(u) \]

and show convergence of the Newton method.

For nearly integrable maps \( h(\theta, \theta') = \frac{(\theta' - \theta)^2}{2} + \varepsilon \Delta h(\theta, \theta') \) we have:

\[ \Delta_\omega v + \varepsilon \left[ (\partial_{11} \Delta h + \partial_{22} \Delta h_-)v + (\partial_{12} \Delta h) v^+ + (\partial_{12} \Delta h^-) v^- \right]. \]

For \( \varepsilon = 0 \) the identity \( u_0(\phi) \equiv \phi \) solves this equation.

It is hard to solve this equation directly. We use Moser’s trick.
Moser’s trick of solving the cohomological equation

Need to solve

\[
DE(u)v = (\partial_{11} h + \partial_{22} h_-)v + \partial_{12} h v^+ + \partial_{12} h^- v^- = -E(u).
\]

Since a solution \( u \) is close to \( u_0(\phi) \equiv \phi \) and \( h(\theta, \theta') \approx \frac{(\theta' - \theta)^2}{2} \), we have

\[
uphi \approx 1, \quad \partial_{11} h \approx 1, \quad \partial_{22} h \approx 1, \quad \partial_{12} h \approx -1, \quad DE(u) \approx \Delta \omega.
\]

Notice that \( DE(u)u_\phi \) is small. Multiply both sides by \( u_\phi \approx 1 \) and add \( v DE(u)u_\phi \).

\[
u_\phi DE(u) v - v DE(u) u_\phi = -u_\phi E(u).
\]

It is possible to rewrite in the form (see Moser-Levi)

\[
\nabla^*_\omega \left( \partial_{12} h \cdot u_\phi \cdot u_\phi \cdot u_\phi^+ \nabla_\omega \left( \frac{v}{u_\phi} \right) \right) = -u_\phi E(u).
\]

Call it a modified homological equation.
Newton’s method (modified)

**Goal:** Suppose $E_\omega(u_0) = E_0$ is small. Using Newton’s method find $E_\omega(u) = 0$ for some $u$ near $u_0$.

Newton Approximation procedure: Expand

$$E(u + v) = E(u) + DE(u)v + Q(v), \quad Q(v) \sim |v|^2 - \text{quadr. error}.$$ 

Solve so-called **homological equation**

$$DE(u)v = -E(u).$$

**Step 1.** Solve $DE(u_0)v_0 + Q'(v_0) = -E(u_0)$. We have for $u_1 = u_0 + v_0$

$$E(u_1) = E(u_0) + DE(u_0)v_0 + Q(v_0) = Q'(v_0) + Q(v_0) \sim E_0^2.$$

**Step n.** Solve $DE(u_n)v_n + Q'(v_n) = -E(u_n)$. We have for $u_{n+1} = u_n + v_n$

$$E(u_{n+1}) = E(u_n) + DE(u_n)v_n + Q(v_n) = Q'(v_n) + Q(v_n) \sim E^2(u_n) \sim E_0^{2n}.$$

The solution $u = u_0 + \sum_{n=1}^{\infty} v_n$ is well-defined as $v_n \to 0$ superexponentially fast in $n$. 
Moser’s trick of solving the homological equation

Need to solve

\[ DE(u)v = (\partial_{11} h + \partial_{22} h^-)v + \partial_{12} h v^+ + \partial_{12} h^- v^- = -E(u). \]

Since a solution \( u \) is close to \( u_0(\phi) \equiv \phi \) and \( h(\theta, \theta') \approx \frac{(\theta' - \theta)^2}{2} \), we have

\[ u_\phi \approx 1, \quad \partial_{11} h \approx 1, \quad \partial_{22} h \approx 1, \quad \partial_{12} h \approx -1, \quad DE(u) \approx \Delta_\omega. \]

Notice that \( DE(u)u_\phi \) is \( \sim v \)-small. Multiply both sides by \( u_\phi \approx 1 \) and add quadratic in \( v \)-term, we have \( v DE(u)u_\phi \).

\[ u_\phi DE(u) v - v DE(u)u_\phi = -u_\phi E(u). \]

It is possible to rewrite in the form (see Moser-Levi)

\[ \Delta_\omega v \approx \nabla_\omega^* \left( \partial_{12} h \cdot u_\phi \cdot u_\phi^+ \nabla_\omega \left( \frac{v}{u_\phi} \right) \right) = -u_\phi E(u). \]
Lemma (Decrease of domain of analyticity)

Let $\omega$ be $(\tau, \gamma)$-diophantine, $g \in W_r$ be zero average $[g] = 0$. Then $\nabla_\omega \psi = g$ has a unique solution $\psi \in W_{r'}$ with $[\psi] = 0$ and $\forall 0 < r' < r$

$$\|\psi\|_{r'} \leq c(\tau, \gamma) \frac{\|g\|_r}{(r - r')^{2+\tau}}.$$

Proof: Expand in Fourier $g(\phi) = \sum_{k \in \mathbb{Z}} g_k \exp(ik\phi)$. The Fourier coefficients of $\psi$ and $g$ are related via

$$\psi_k = \frac{g_k}{\exp(2\pi k\omega) - 1} \neq 0, \quad \psi_0 = 0.$$

By diophantine condition $|\exp(2\pi k\omega) - 1| \geq c(\gamma)|k|^{-1-\tau}$ for all $k \neq 0$. Since $g \in W_r$, we have $|g_k| \leq \|g\|_r \exp(-2\pi |k|r)$ for all $k \in \mathbb{Z}$. So

$$|\psi_k| \leq \frac{\|g\|_r}{\exp(2\pi |k|r)} |k|^{1+\tau} c^{-1}(\gamma) = \frac{\|g\|_r}{\exp(2\pi |k|s)} \times \frac{|k|^{1+\tau} c^{-1}(\gamma)}{\exp(2\pi |k|(r - s))}.$$
We have

\[ |\psi_k| \leq \frac{\|g\|_r \exp(2\pi |k| s) \times |k|^{1+\tau} c^{-1}(\gamma)}{\exp(2\pi |k| (r - s))}. \]

Notice that for all \( n \) we have

\[ x \exp(-x) \leq 1/e \implies \exp(-a|n|)|n|^b \leq \exp(-b)(b/a)^b. \]

Substitute \( a = 2\pi (r - s) \) and \( b = 1 + \tau \) and get

\[ |\psi_k| \leq \frac{\|g\|_r \exp(2\pi |k| s)}{\exp(2\pi |k| (r - s))} \times \frac{c(\gamma, \tau)}{(r - s)^{1+\tau}}. \]

Finally,

\[ \|\psi\|_{r'} \leq \sum_k |\psi_k| \exp(2\pi |k| r') \leq \frac{c(\gamma, \tau)\|g\|_r}{(r - s)^{1+\tau}} \times (1 - \exp(2\pi (s - r')))^{-1} \leq \frac{c(\gamma, \tau)\|g\|_r}{(r - s)^{1+\tau}(s - r')} \cdot \]

Using \( s = (r + r')/2 \) completes the proof.
Lemma (Existence of a solution to a homological equation)

Let \( \omega \) be \((\tau, \gamma)\)-diophantine, \( u \in W_r \) and close to \( u_0(\phi) \equiv \phi \). Then there is a unique solution \( v \) to the modified homological equation

\[
\nabla^* \omega \left( \partial_{12} h u_\phi u_\phi^+ \nabla \omega \left( \frac{v}{u_\phi} \right) \right) = -u_\phi E(u) \quad \text{s.t.}
\]

\[
\|v\|_{r'} \leq c(\tau, \gamma) \frac{\|E(u)\|_r}{(r - r')^{2(2+\tau)}}.
\]

Proof (see Moser-Levi, Lemma 8) Use the fact that

\[
u_\phi \approx 1, \quad \partial_{11} h \approx 1, \quad \partial_{22} h \approx 1, \quad \partial_{12} h \approx -1.
\]

We get that small deviation from

\[
\nabla^*_\omega \nabla_\omega v = -u_\phi E(u).
\]

To solve it apply previous lemma twice.
Convergence of the approximation procedure

Lemma

Let $\omega$ be $(\tau, \gamma)$-diophantine, $u$ be near $u_0$ in $W_r$, $v$ be a unique solution to the modified homological equation. Then

$$\|E(u + v)\|_\rho \leq \frac{c}{(r - \rho)^2} \|E(u)\|^2_r. \quad (4)$$

Choose $r_n = r_\infty + 2^{-n}(r_0 - r_\infty)$ with $r_n \to r_\infty$. Construct a sequence $u_0, u_1, \ldots, u_n, \ldots$ by $u_{n+1} = u_n + v_n$, where $v_n \in W_{r_n}$ solved the homological equation.

Let $\varepsilon_n = \|E(u_n)\|_{r_n}$. Then $\|v_n\|_{r_n} \leq \frac{c\varepsilon_n}{(r_n - r_{n+1})^{2(2+\tau)}}$.

Therefore, for $a = 2^{2(2+\tau)}$ we have $\varepsilon_{n+1} \leq ca^n\varepsilon_n^2$ and

$$\|u_n - u_0\|_{r_n} \leq \sum_{k=0}^{n-1} \|v_k\|_{r_k} \leq \sum_{k=0}^{n-1} \frac{c \varepsilon_n}{(r_n - r_{n+1})^{2(2+\tau)}} \leq \frac{2ac \varepsilon_0}{(r_0 - r_\infty)^{2(2+\tau)}}.$$
Lemma

Let $\omega$ be $(\tau, \gamma)$-diophantine, $u$ be near $u_0$ in $W_r$, $v$ be a unique solution to the modified homological equation. Then

$$\|E(u + v)\|_\rho \leq \frac{c}{(r - \rho)^2} \|E(u)\|_r^2.$$  \hspace{1cm} (4)

Choose $r_n = r_\infty + 2^{-n}(r_0 - r_\infty)$ with $r_n \to r_\infty$. Construct a sequence $u_0, u_1, \ldots, u_n, \ldots$ by $u_{n+1} = u_n + v_n$, where $v_n \in W_{r_n}$ solved the homological equation.

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$$\|u_n - u_0\|_{r_n} \leq \sum_{k=0}^{n-1} \|v_k\|_{r_k} \leq \sum_{k=0}^{n-1} \frac{c}{(r_n - r_{n+1})^{2(2+\tau)}} \varepsilon_n \leq \frac{2ac \varepsilon_0}{(r_0 - r_\infty)^{2(2+\tau)}}.$$  \hspace{1cm} (5)
Convergence of the approximation procedure

**Lemma**

Let $\omega$ be $(\tau, \gamma)$-diophantine, $u$ be near $u_0$ in $W_r$, $v$ be a unique solution to the modified homological equation. Then

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Therefore, for $a = 2^{2(2+\tau)}$ we have $\varepsilon_{n+1} \leq c a^n \varepsilon_n^2$ and

$$\|u_n - u_0\|_{r_n} \leq \sum_{k=0}^{n-1} \|v_k\|_{r_k} \leq \sum_{k=0}^{n-1} \frac{c \varepsilon_n}{(r_n - r_{n+1})^{2(2+\tau)}} \leq \frac{2ac \varepsilon_0}{(r_0 - r_\infty)^{2(2+\tau)}}.$$
Convergence of the approximation procedure

Lemma

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$$
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$$\|E(u + v)\|_\rho \leq \frac{c}{(r - \rho)^2} \|E(u)\|_r^2.$$  \hfill (5)

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$$\|u_n - u_0\|_{r_n} \leq \sum_{k=0}^{n-1} \|v_k\|_{r_k} \leq \sum_{k=0}^{n-1} \frac{c\varepsilon_n}{(r_n - r_{n+1})^{2(2+\tau)}} \leq \frac{2ac\varepsilon_0}{(r_0 - r_\infty)^{2(2+\tau)}}.$$
Finitely differentiable KAM theorem

**Lemma**

Let $\omega$ be $(\tau, \gamma)$-diophantine, $u$ be near $u_0$ in $W_r$, $v$ be a unique solution to the modified homological equation. Then

$$\|E(u + v)\|_\rho \leq \frac{c}{(r - \rho)^2} \|E(u)\|_r^2.$$  \hfill (5)

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Lemma

Let $\omega$ be $(\tau, \gamma)$-diophantine, $u$ be near $u_0$ in $W_r$, $v$ be a unique solution to the modified homological equation. Then

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$$

Choose $r_n = r_\infty + 2^{-n}(r_0 - r_\infty)$ with $r_n \to r_\infty$. Construct a sequence $u_0, u_1, \ldots, u_n, \ldots$ by $u_{n+1} = u_n + v_n$, where $v_n \in W_{r_n}$ solved the homological equation.

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Therefore, for $a = 2^{2(2+\tau)}$ we have $\varepsilon_{n+1} \leq ca^n \varepsilon_n^2$ and

$$
\|u_n - u_0\|_{r_n} \leq \sum_{k=0}^{n-1} \|v_k\|_{r_k} \leq \sum_{k=0}^{n-1} \frac{c \varepsilon_n}{(r_n - r_{n+1})^{2(2+\tau)}} \leq \frac{2ac \varepsilon_0}{(r_0 - r_\infty)^{2(2+\tau)}}.
$$
Finitely differentiable KAM theorem

Lemma

Let \( \omega \) be \((\tau, \gamma)\)-diophantine, \( u \) be near \( u_0 \) in \( W_r \), \( v \) be a unique solution to the modified homological equation. Then

\[
\| E(u + v) \|_\rho \leq \frac{c}{(r - \rho)^2} \| E(u) \|^2_r. \tag{5}
\]

Choose \( r_n = r_\infty + 2^{-n}(r_0 - r_\infty) \) with \( r_n \to r_\infty \). Construct a sequence \( u_0, u_1, \ldots, u_n, \ldots \) by \( u_{n+1} = u_n + v_n \), where \( v_n \in W_{r_n} \) solved the homological equation.

Let \( \varepsilon_n = \| E(u_n) \|_{r_n} \). Then

\[
\| v_n \|_{r_n} \leq \frac{c \varepsilon_n}{(r_n - r_{n+1})^{2(2 + \tau)}}.
\]

Therefore, for \( a = 2^{2(2 + \tau)} \) we have

\[
\varepsilon_{n+1} \leq c a^n \varepsilon_n^2
\]

and

\[
\| u_n - u_0 \|_{r_n} \leq \sum_{k=0}^{n-1} \| v_k \|_{r_k} \leq \sum_{k=0}^{n-1} \frac{c \varepsilon_n}{(r_n - r_{n+1})^{2(2 + \tau)}} \leq \frac{2ac \varepsilon_0}{(r_0 - r_\infty)^{2(2 + \tau)}}.
\]
Finitely differentiable KAM theorem

Lemma

Let \( \omega \) be \((\tau, \gamma)\)-diophantine, \( u \) be near \( u_0 \) in \( W_r \), \( v \) be a unique solution to the modified homological equation. Then

\[
\| E(u + v) \|_{\rho} \leq \frac{c}{(r - \rho)^2} \| E(u) \|_{r}^2. \tag{5}
\]

Choose \( r_n = r_\infty + 2^{-n}(r_0 - r_\infty) \) with \( r_n \to r_\infty \). Construct a sequence \( u_0, u_1, \ldots, u_n, \ldots \) by \( u_{n+1} = u_n + v_n \), where \( v_n \in W_{r_n} \) solved the homological equation.

Let \( \varepsilon_n = \| E(u_n) \|_{r_n} \). Then \( \| v_n \|_{r_n} \leq \frac{c \varepsilon_n}{(r_n - r_{n+1})^{2(2+\tau)}} \).

Therefore, for \( a = 2^{2(2+\tau)} \) we have \( \varepsilon_{n+1} \leq ca^n \varepsilon_n^2 \) and

\[
\| u_n - u_0 \|_{r_n} \leq \sum_{k=0}^{n-1} \| v_k \|_{r_k} \leq \sum_{k=0}^{n-1} \frac{c \varepsilon_n}{(r_n - r_{n+1})^{2(2+\tau)}} \leq \frac{2ac \varepsilon_0}{(r_0 - r_\infty)^{2(2+\tau)}}. \]
We define the norm $\|f\|_{C^\ell} := \sum_{|\alpha| \leq \ell} |\partial^\alpha f|_{C^\mu}$ for $\mu := \ell - [\ell] < 1$. We also use the standard abbreviation $\alpha! := \alpha_1! \cdots \alpha_n!$ and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. For $0 < \mu < 1$, $|f|_{C^\mu}$ is the standard Hölder norm.

Suppose $H_\varepsilon$ (resp. a generating function $h_\varepsilon$) is $C^\ell$ smooth.

Suppose $H_\varepsilon^n$ (resp. a generating function $h_\varepsilon^n$) be a sequence of analytic Hamiltonians approximating $H_\varepsilon$ (resp. $h_\varepsilon$).

Apply analytic KAM to each $H_\varepsilon^n$ (resp. $h_\varepsilon^n$). Obtain a solution $u_n$.

Obtain estimates on speed of convergence of $u_n$ and conclude smoothness of $u$. 
Approximation lemmas

If $f$ is $C^\ell$ smooth, then it can be approximated with $\ell$ dependent speed.

Lemma

There is a family of convolution operators

\[ S_r f(x) = r^{-n} \int_{\mathbb{R}^n} K(r^{-1}(x - y))f(y)\,dy, \quad 0 < r \leq 1, \]

from $C^0(\mathbb{R}^n)$ into the space of entire functions on $\mathbb{R}^n$ with the following property. For every $d \geq \ell \geq 0$, there exists a constant $c(\ell, n, d) > 0$ such that, for every $f \in C^\ell(\mathbb{R}^n)$ we have

\[ \| S_r f - f \|_{C^s} \leq c \| f \|_{C^\ell} r^{\ell-s}, \quad s \leq \ell, \]

\[ \| S_r f \|_{C^s} \leq c \| f \|_{C^\ell} r^{\ell-s}, \quad \ell \leq s \leq d. \]
Approximation lemmas

Depending on speed of approximations of \( f \) by analytic functions \( f_n \)
one can obtain smoothness of \( f \).

**Lemma**

*Assume that \( f : \mathbb{R}^n \to \mathbb{R} \) is the limit of a sequence of real analytic functions \( f_k(x) \) defined in the complex strip \( |\Im x| < r_k = 2^{-k}, \ x \in \mathbb{C}^n \), with \( 0 < r_0 \leq 1 \) and

\[
|f_k(x) - f_{k-1}(x)| \leq A r_k^l, \ \forall x, \ |\Im x| \leq r_k.
\]

Then \( f \in C^s(\mathbb{R}^n) \) for every \( s \leq l \) which is not an integer and, moreover,

\[
|f(x) - f_0(x)|_{C^s} \leq c A(\theta(1 - \theta)) r_0^l
\]

for \( 0 < \theta = s - [s] < 1 \) and a suitable constant \( c = c(l, n) > O. \)