DYNAMICAL SPECTRAL RIGIDITY AMONG $\mathbb{Z}_2$-SYMMETRIC STRICTLY CONVEX DOMAINS CLOSE TO A CIRCLE

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Abstract. We show that any sufficiently (finitely) smooth $\mathbb{Z}_2$-symmetric strictly convex domain sufficiently close to a circle is dynamically spectrally rigid, i.e. any deformation among domains in the same class which preserves the length of all periodic orbits must necessarily be an isometric deformation.

This is a preliminary version

1. Introduction

In this paper we study questions motivated by the famous question of M. Kac [11]: “Can one hear the shape of a drum?” Formally, given a domain $\Omega \subset \mathbb{R}^2$, consider the spectrum of the Laplace operator in $\Omega$ with (e.g.) Dirichlet boundary condition\(^1\), denote by $\Delta(\Omega)$, i.e. the set of $\lambda$ so that

$$\Delta u + \lambda^2 u = 0, \quad u = 0 \text{ on } \partial \Omega.$$  

“Does the Laplace spectrum determine a domain?” There is a number of counterexamples to this question (see e.g. [8, 21, 22]). Notably Sunada [21] constructed isospectral sets of arbitrary large cardinality. However, the domains considered in such examples are neither smooth nor convex.

For planar domains with a $C^\infty$ smooth boundary Osgood-Phillips-Sarnak [13, 14, 15] showed that an isospectral set of planar domains is compact in $C^\infty$ topology. Sarnak [20] conjectured that an isospectral set consists of isolated points, namely, any planar domain $\Omega$ with $C^\infty$ smooth boundary cannot be perturbed to an isospectral domain $\Omega'$ unless perturbation is an isometry.

In the affirmative direction Hezari–Zelditch [10] proved that given an ellipse $\mathcal{E}$, any one-parameter $C^\infty$-deformation $\Omega_\tau$ which preserves

\(^1\) From the physical point of view, the Dirichlet eigenvalues $\lambda$ correspond to the eigenfrequencies of a vibrating membrane of shape $\Omega$ which is fixed along its boundary.
the Laplace spectrum (with respect to either Dirichlet or Neumann boundary conditions) and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group of the ellipse has to be flat (i.e., all derivatives have to vanish for $\tau = 0$). Popov–Topalov [19] extended these results (see also [18]). Further historical remarks on the inverse spectral problem can also be found in [10].

1.1. The length spectrum and relation with the Laplace spectrum. An object closely related to the Laplace spectrum is so-called length spectrum. The **Length Spectrum** of a domain $\Omega$ is the set:

$$\mathcal{L}(\Omega) = \mathbb{N}\{\text{length of closed geodesic of } \Omega\} \cup \mathbb{N}\{l_{\partial \Omega}\},$$

where $l_{\partial \Omega}$ denotes the length of the boundary $\partial \Omega$ and $\mathbb{N} = \{1, 2, \cdots \}$.

There is a remarkable relation between the Laplace spectrum and the length spectrum for convex domains.

Andersson–Melrose (see [1, Theorem (0.5)], which substantially generalizes some earlier result by [3, 7]) proved that, for strictly convex $C^\infty$ domains, the following relation between the wave trace and the length spectrum holds:

$$\text{sing sup} \left( t \mapsto \sum_{\lambda_j \in \Delta(\Omega)} \exp(i\lambda_j t) \right) \subset \pm \mathcal{L}(\Omega) \cup \{0\}. \quad (1.1)$$

Generically the above inclusion becomes an equality and the Laplace spectrum determines the length spectrum (see Remark 2.9).

1.2. Inverse Dynamical Problem. The length spectrum of $\Omega$ is clearly invariant under isometries: it is then natural to ask the following: Can $\Omega$ be determined by $\mathcal{L}(\Omega)$ (modulo isometries)?

The above question turns out to be quite difficult to answer; there are counterexamples as soon as one drops the convexity assumptions (see below), but in case of convex sufficiently smooth domain it is an open problem. We present results on a deformational version of this question which we call dynamical spectral rigidity.

1.3. Dynamical spectral rigidity. We call a $C^1$ smooth one-parameter family $(\Omega_\tau)_{\tau \in U}$ dynamically isospectral if $\mathcal{L}(\Omega_\tau) = \mathcal{L}(\Omega_0)$ for any $\tau \in U$. A domain is called dynamical spectrally rigid if any dynamically isospectral family $(\Omega_\tau)_{\tau \in U}$ is a family of isometries.

Consider the space $\mathcal{S}^r$ of strictly convex domains with axial symmetry whose boundary is $C^{r+1}$ and $r$ is sufficiently large.
We prove that any domain $\Omega \in S^r$ with nearly circular boundary is dynamical spectrally rigid.\footnote{A more precise formulation will be provided in the next section.}

1.4. Local rigidity. Local rigidity problems, i.e. isospectral deformations, were considered in some early works on variations of the spectral functions and wave invariants. It is worth to mention the work of Guillemin-Kazhdan \cite{9}. They introduce the following definition of (deformation) spectral rigidity.

Let $(M, g)$ be a compact boundaryless Riemannian manifold. A family $(g_\tau)_{\tau \in U}$ of Riemannian metrics on $M$ smoothly depending on the parameter $\tau \in [-\varepsilon, \varepsilon]$ is called the deformation of the metric $g$ if $g_0 = g$. A deformation is called trivial if there exists a one-parameter family of diffeomorphisms $\varphi_\tau : M \to M$ such that $\varphi_0 = Id$, and $g_\tau = (\varphi_\tau)^* g_0$. Given a deformation $g_\tau (|\tau| \leq \varepsilon)$, let $\Delta_\tau : C^\infty(M) \to C^\infty(M)$ be the Laplace-Beltrami operator corresponding to the metric $g_\tau$. The deformation is called isospectral if, for all $|\tau| \leq \varepsilon$ spectra of the operators $\Delta_\tau$ and $\Delta_0$ coincide (counting multiplicities). A Riemannian manifold $(M, g)$ is called deformationally spectrally rigid if it does not admit non-trivial isospectral deformations.

They show that a negatively curved surface can’t be isospectrally deformed among negatively curved surfaces. It is shown that if $\gamma$ a closed geodesic of a metric $g$ that is non-degenerate and of multiplicity one in the length spectrum, then for any deformation $g_\tau$ of $g$, $\gamma$ deforms smoothly as a closed geodesic $\gamma_\tau$ of $g_\tau$, then the first variation of the length $L_\gamma$ is given by $\int_\gamma \dot{g} ds$. For a negative curved surface, there does not exist a non-trivial symmetric tensor $\dot{g}$ satisfying the isospectral condition that $\int_\gamma \dot{g} ds = 0$. This result for compact manifolds of negative curvature was proven in \cite{5}

A global version of the \cite{9} result is to Croke \cite{4} and Otal \cite{16}. Namely, a properly defined Marked Length spectrum determines a surface of negative curvature up to an isometry.

Our result is an analogue of \cite{9} and \cite{5} for $Z_2$ symmetric billiard table close to the disk. (A natural analogue of negative curved surfaces is hyperbolic billiard table).

Another example of deformational spectral rigidity appears in De la Llave, Marco and Moriyón \cite{6}. It is shown that there are no non-trivial deformations of exact symplectic mappings $B_\tau$, $\tau \in [0, 1]$, leaving the length spectrum fixed when $B_\tau$ are Anosov’s mappings on a symplectic manifold. One of the reasons for symplectic rigidity in \cite{6} is that all periodic points of $B_\tau$ are hyperbolic and form a dense set.
Now we turn to a rigorous statement of our results and their proof.

2. Statement of results

We denote by $\mathcal{D}^r$ the set of strictly convex planar domains $\Omega$ whose boundary is $C^{r+1}$ smooth and by $\mathcal{S}^r \subset \mathcal{D}^r$ the subset of domains with axial symmetry.

**Definition 2.1.** A closed geodesic in $\Omega$ is a (not necessarily convex) polygon inscribed in $\partial \Omega$ so that at each vertex, the angles formed by each of the two sides joining at the vertex with the tangent line to $\partial \Omega$ are equal. The perimeter of the polygon is called the length of the geodesic.

**Definition 2.2.** The Length Spectrum$^4$ of a domain $\Omega$ is the set:

$$\mathcal{L}(\Omega) = \mathbb{N}\{\text{length of closed geodesic of } \Omega\} \cup \mathbb{N}\{l_{\partial \Omega}\},$$

where $l_{\partial \Omega}$ is the length of the boundary $\partial \Omega$.

Obviously, the length spectrum of $\Omega$ is invariant under isometries: it is then natural to ask the following

**Question** (Inverse Dynamical Problem). Can $\Omega \in \mathcal{D}^r$ be determined by $\mathcal{L}(\Omega)$ (modulo isometries)?

Our results on a deformational version of this question which we call dynamical spectral rigidity. In order to state this condition we need some preparation.

**Definition 2.3.** We say that $(\Omega_\tau)_{|\tau| \leq \varepsilon}$ is a $C^1$ one-parameter family of domains in $\mathcal{S}^r$ if $\Omega_\tau \in \mathcal{S}^r$ for any $|\tau| \leq \varepsilon$ and there exists

$$\gamma(\tau, s) : U \times \mathbb{T}^1 \to \mathbb{R}^2$$

so that $\gamma(\tau, s)$ is continuously differentiable in $\tau$ with bounded derivative and, for any $\tau \in U$, the map $\gamma(\tau, \cdot)$ is a $C^{r+1}$ diffeomorphism of $\mathbb{T}^1$ onto $\partial \Omega_\tau$.

**Remark 2.4.** Notice that for a given family $(\Omega_\tau)_{|\tau| \leq \varepsilon}$, the choice of $\gamma$ is quite arbitrary: given any family of $C^{r+1}$ circle diffeomorphisms $s(\tau, \sigma)$ we can find another parametrization of the same family of domains by choosing $\tilde{\gamma}(\tau, \sigma) = \gamma(\tau, s(\tau, \sigma))$.

$^3$ We denote the superscript with $r$, because the associated billiard map is $C^r$ (see the next section).

$^4$ It possible to give an alternative definition of the length spectrum:

$$\mathcal{L}(\Omega) = \mathbb{N}\{\text{length of closed geodesic of } \Omega\}.$$
Definition 2.5. A family \((\Omega_\tau)_{|\tau| \leq \varepsilon}\) is said to be isometric (or trivial) if there exists a family \((\mathcal{I}_\tau)_{\tau \in U}\) of isometries \(\mathcal{I}_\tau : \mathbb{R}^2 \to \mathbb{R}^2\) (i.e. composition of a rotation and a translation) so that \(\Omega_\tau = \mathcal{I}_\tau \Omega_0\).

A question related to the inverse dynamical problem is whether or not, given a domain \(\Omega\), there exist so called *isospectral deformations* of \(\Omega\), which we now define.

Definition 2.6. A family \((\Omega_\tau)_{|\tau| \leq \varepsilon}\) is dynamically isospectral (or length-isospectral) if \(L(\Omega_\tau) = L(\Omega_0)\) for any \(\tau \in U\).

Definition 2.7. A domain \(\Omega_0 \in S^r\) is said to be dynamically spectrally rigid (resp. \(S^r\)-dynamically spectrally rigid) if any dynamically isospectral family of deformations \((\Omega_\tau)_{\tau \in U} \subset D^r\) (resp. \((\Omega_\tau)_{\tau \in U} \subset S^r\)) is an isometric family.

Recall that a closed curve is a circle if and only if its radius of curvature is constant.

Definition 2.8. A domain \(\Omega \in D^r\) of perimeter 1 is said to be \(\delta\)-close to a circle if
\[
\|2\pi \rho(s) - 1\|_{C^{r-1}} \leq \delta.
\]

Recall that if \(\Omega \in D^r\), \(\partial \Omega\) is \(C^{r+1}\), thus its curvature is \(C^{r-1}\). A domain \(\Omega\) of arbitrary perimeter is said to be \(\delta\)-close to a circle if its rescaling of perimeter 1 is \(\delta\)-close to a circle.

We are now able to state the main result of this paper.

**Main Theorem.** Let \(r\) be sufficiently large and \(\Omega \in S^r\), whose boundary \(\partial \Omega\) is sufficiently close to a circle, then \(\Omega\) is \(S^r\)-dynamically spectrally rigid.

**Remark 2.9.** It turns out that the Laplace spectrum generically determines the length spectrum, namely, in (1.1) we have an equality (see [17]). Generic conditions are as follows:

1. no two distinct orbits have the same length
2. the Poincaré map of any periodic orbit of the associated billiard map (see (3.1)) is non-degenerate.

It is convenient to restate the result as follows:

Definition 2.10. A domain \(\Omega \in S^r\) is said to be normalized if the marked point of \(\partial \Omega\) is at the origin of \(\mathbb{R}^2\), its symmetry axis coincides with the \(x\)-axis and \(\Omega\) lies in the right half plane. A family \((\Omega_\tau)_{\tau \in U}\) is said to be normalized if \(\Omega_\tau\) is normalized for any \(\tau \in U\).
Given a family \((\Omega_\tau)_{\tau \in U}\), we can always construct a normalized family \((\tilde{\Omega}_\tau)_{\tau \in U}\) by isometries as follows:

- translate the domain so that the marked point of \(\partial \Omega_\tau\) is at the origin of \(\mathbb{R}^2\).
- rotate the domain around the origin so that the symmetry axis is the \(x\)-axis and the domain lies in the right half plane.

By smoothness (in \(\tau\)) of \((\Omega_\tau)_{\tau \in U}\), we gather that \((\tilde{\Omega}_\tau)_{\tau \in U}\) can be chosen to be as smooth as the original family \((\Omega_\tau)_{\tau \in U}\). We call \((\tilde{\Omega}_\tau)_{\tau \in U}\) the normalization of the family \((\Omega_\tau)_{\tau \in U}\). Observe that, since \(\tilde{\Omega}_\tau\) is obtained from \(\Omega_\tau\) via an isometry, we have \(L(\Omega_\tau) = L(\tilde{\Omega}_\tau)\); in particular, if \((\Omega_\tau)_{\tau \in U}\) is a dynamically isospectral family, so is \((\tilde{\Omega}_\tau)_{\tau \in U}\).

Our Main Theorem can then be restated as follows:

**Theorem 2.11.** Let \(r\) be sufficiently large and \((\Omega_\tau)_{\tau \in U}\) be a normalized \(C^1\)-family of domains in \(S^r\) sufficiently \(C^6\)-close to a circle. If \(L(\Omega_\tau) = L(\Omega_0)\) for all \(\tau \in U\), then \(\Omega_\tau = \Omega_0\).

From now on we will assume that \((\Omega_\tau)_{\tau \in U}\) is a normalized \(C^1\)-family of \(S^r\)-domains so that \(\Omega_0\) has perimeter 1.

Elementary ODE considerations imply that there exists a parametrization \(\gamma : U \times T^1 \to \mathbb{R}\) of \((\Omega_\tau)_{\tau \in U}\) which satisfies the following properties:

1. \(\gamma(0, s)\) is the arc-length parametrization of \(\partial \Omega_0\)
2. for any \(\tau \in U\), the point \(\gamma(\tau, 0)\) is the marked point of \(\partial \Omega_\tau\), that is, the origin of \(\mathbb{R}^2\).
3. for any \(\tau \in U\) and \(s \in T^1\), the vector \(\partial_\tau \gamma(\tau, s)\) is orthogonal to \(\partial \Omega_\tau\) at \(\gamma(\tau, s)\).

We call this parametrization the Canonical parametrization of the family \((\Omega_\tau)_{\tau \in U}\). Then we define the orthogonal deformation associated to \((\Omega_\tau)_{\tau \in U}\) to be the function \(n(\tau, s)\) which satisfies \(\partial_\tau \gamma(\tau, s) = n(\tau, s)N(\tau, s)\), where \(N(\tau, s)\) is the outgoing unit normal vector to \(\partial \Omega_\tau\) at \(\gamma(\tau, s)\).

**Remark 2.12.** Observe that if \((\Omega_\tau)_{\tau \in U} \subset S^r\) is a normalized family, it is generated by orthogonal deformations \(n(\tau, s)\) that satisfy the following properties: for any \(\tau \in U\):

1. \(n(\tau, \cdot)\) is an even function, i.e. \(n(\tau, s) = n(\tau, -s)\);
2. \(n(\tau, 0) = 0\).

We can thus further restate our Main Theorem as

**Theorem 2.13.** Let \(r\) be sufficiently large and \((\Omega_\tau)_{\tau \in U}\) be a normalized \(C^1\)-family of domains in \(S^r\) sufficiently close to a circle and \(n\) the associated normal perturbation. If \(L(\Omega_\tau) = L(\Omega_0)\) for all \(\tau \in U\), then \(n = 0\).
3. Billiard dynamics of axially symmetric domains

Let \( \Omega \in S^r \) be a domain with an axis of symmetry; for definiteness we fix the length of its boundary to be equal to 1 and the counterclockwise orientation on \( \partial \Omega \) to be the positive orientation. We mark a point \( P_0 \in \partial \Omega \) to be one of the two points that lie on the symmetry axis of \( \Omega \); we will consider the point \( P_0 \) to be the origin of all our parametrizations of \( \partial \Omega \). In particular, since \( \Omega \) has perimeter 1, the other intersection of \( \partial \Omega \) with the symmetry axis corresponds to the arc-length parameter \( 1/2 \). We consider the billiard dynamics on \( \Omega \), which is described as follows: a point particle travels with constant velocity in the interior of \( \Omega \); when the particle hits \( \partial \Omega \), it is reflected according to the law of optical reflection: angle of incidence equals angle of reflection. Closed geodesics of \( \Omega \) are, therefore, in 1-to-1 correspondence to periodic trajectories of the billiard dynamics. It is customary to study the dynamics by passing to a discrete-time version of it, i.e. to a map on the canonical Poincaré section \( M = \partial \Omega \times [-1,1] \). The first variable (which is usually parametrized in arc-length, denoted with \( s \in \mathbb{T}_1 = \mathbb{R}/\mathbb{Z} \)) indicates the point at which the particle has collided with \( \partial \Omega \) and the second coordinate \( \Phi \) equals \( \cos(\varphi) \), where \( \varphi \) is the angle that the outgoing trajectory forms with the positively oriented tangent to \( \partial \Omega \). The billiard ball map on \( M \) is then defined as

\[
\begin{align*}
    f : \partial \Omega \times [-1,1] &\rightarrow \partial \Omega \times [-1,1] \\
    (s, \Phi) &\mapsto (s', \Phi')
\end{align*}
\]

where \( s' \) is the coordinate of the point at which the trajectory emanating from \( s \) with angle \( \varphi \) collides once again with \( \partial \Omega \) and \( \Phi' = \cos(\varphi') \), where \( \varphi' \) is the angle of incidence of the new collision with the positively oriented tangent to \( \partial \Omega \) at \( s' \). The map \( f \) is an exact twist diffeomorphism which preserves the area form \( ds \wedge d\Phi \). Let us denote by

\[
L(s, s') = \|\gamma(s) - \gamma(s')\|
\]

the Euclidean distance between two points on \( \partial \Omega \). Then \( L \) is a generating function of the billiard ball map, i.e.

\[
\begin{align*}
    \frac{\partial L}{\partial s}(s, s') &= -\Phi \\
    \frac{\partial L}{\partial s'}(s, s') &= \Phi'.
\end{align*}
\]

Having defined the billiard map \( f \) we proceed to prove the following

Lemma 3.1. The length spectrum of \( \Omega \) has zero Lebesgue measure.

Proof. Recall that by Sard’s Lemma the set of critical values of a real valued \( C^r \)-function defined on an \( n \)-dimensional manifold has zero
Lebesgue measure provided that \( r \geq n \). For any \( q \), let us define the function

\[
L_q(s, \Phi) = L(s_1, s_0) + L(s_2, s_1) + \cdots + L(s_0, s_{q-1})
\]

where \((s_k, \Phi_k) = f^k(s, \Phi)\). Periodic orbits of period \( q \) of the billiard map correspond to critical points of \( L_q \). Indeed, the law angle of reflection at \( s_k \) equals to the angle of incidence implies that partial derivative of \( L_q \) with respect to \( s_k \) equals zero. Therefore, their lengths correspond to critical values of \( L_q \). Since the billiard map \( f \) is \( C^r \) with \( r > 2 \), we conclude that the set of lengths of periodic orbits of period \( q \) has zero Lebesgue measure; by taking the (countable) union over \( q \) we obtain \( \mathcal{L}(\Omega) \) and we conclude that \( \mathcal{L}(\Omega) \) has zero Lebesgue measure. \( \square \)

Consider a periodic orbit of period \( q \); let \( p \in \mathbb{Z} \) be its winding number (i.e. the number of times that the associated polygon wraps around the boundary \( \partial \Omega \)); then the rotation number of the orbit is defined as the ratio \( p/q \). The following lemma is a simple consequence of the fact that \( \Omega \) has axial symmetry and of the normalization that we have chosen.

**Lemma 3.2.** For any \( q > 1 \), there exists a periodic orbit, which we denote with \( S^q(\Omega) \) and call marked symmetric maximal periodic orbit, of rotation number \( 1/q \) and maximal length passing through the origin \( s = 0 \) in \( \partial \Omega \).

**Proof.** We distinguish the cases of even and odd period.

**Case 1:** \( q = 2k \) is even. We claim there exists a \( q \)-periodic orbit passing through \( s = 0 \) and \( s = 1/2 \). Indeed, let us fix \( s_0 = 0 \) and \( s_k = 1/2 \), and consider the problem of maximizing the function

\[
L^q(\xi) := 2 \sum_{i=0}^{k-1} L(s_i, s_{i+1})
\]

on the compact set \( 0 = s_0 \leq s_1 \leq \cdots \leq s_{k-1} \leq s_k = 1/2 \), where \( \xi = (s_1, \ldots, s_{k-1}) \). By the triangle inequality and strict convexity, the maximum is attained at a critical point \((\bar{s}_1, \ldots, \bar{s}_{k-1})\) so that \( s_0 < s_1 < \cdots < s_{k-1} < s_k \). If we fix conventionally \( \bar{s}_0 = 0, \bar{s}_k = 1/2 \), we have

\[
\partial_1 L(\bar{s}_i, \bar{s}_{i+1}) = -\partial_2 L(\bar{s}_{i-1}, \bar{s}_i), \quad i = 1, \ldots, k-1.
\]

Completing \( \bar{s}_{2k-i} = -\bar{s}_i, i = 1, \ldots, k-1 \), we obtain a periodic orbit of period \( 2k = q \), which is of maximal length among symmetric orbits.

**Case 2:** \( q = 2k + 1 \) is odd. We claim there exists a periodic orbit passing through \( \gamma(s = 0) \) and so that the segment \( \gamma(s_k)\gamma(s_{k+1}) \) is
perpendicular to the symmetry axis. Indeed, let us fix \( s_0 = 0 \) and consider the problem of maximizing the function
\[
L^q(\xi) := \sum_{i=0}^{k-1} 2L(s_i, s_{i+1}) + L(s_k, -s_k)
\]
on the compact set \( 0 = s_0 \leq s_1 \leq \cdots \leq s_k \leq 1/2 \), where \( \xi = (s_1, \cdots, s_k) \). Once again by the triangle inequality and strict convexity, the maximum is attained at a critical point \((\bar{s}_1, \ldots, \bar{s}_k)\) so that
\[
0 < \bar{s}_1 < \cdots < \bar{s}_{k-1} < 1/2.
\]
Moreover, if we fix conventionally \( \bar{s}_0 = 0 \), we have
\[
0 = \partial_1 L(\bar{s}_i, \bar{s}_{i+1}) + \partial_2 L(\bar{s}_{i-1}, \bar{s}_i), \quad i = 1, \ldots, k - 1
\]
\[
0 = \partial_2 L(\bar{s}_{k-1}, \bar{s}_k) + \frac{1}{2} \partial_1 L(\bar{s}_k, -\bar{s}_k) - \frac{1}{2} \partial_2 L(\bar{s}_k, -\bar{s}_k)
\]
\[
= \partial_2 L(\bar{s}_{k-1}, \bar{s}_k) + \partial_1 L(\bar{s}_k, -\bar{s}_k)
\]
Completing \( \bar{s}_{2k+1-i} = -\bar{s}_i, i = 1, \ldots k - 1 \), we obtain a periodic orbit of period \( 2k + 1 = q \) which is of maximal length amongst all symmetric orbits. □

Remark 3.3. In the previous lemma we found symmetric periodic orbits as extremal points of a suitably defined variational problem. In general such orbits are not uniquely determined\(^5\). Even if it is possible to show that for sufficiently small perturbations of circles, such orbits are uniformly non-degenerate (i.e. by the implicit function theorem they all persist for sufficiently small perturbations of \( \Omega \)), we will not need non-degeneracy of such orbits to conclude our argument. However, for any \( q \) and \( \tau \in U \), we will fix once and for all a marked symmetric maximal periodic orbit which we denote with \( S^q(\Omega_\tau) \).

Denote \( \gamma_\tau(s) = \gamma(\tau, s) \) and \( L_\tau(s, s') = \|\gamma_\tau(s) - \gamma_\tau(s')\| \). For \( q \geq 2 \), let \( L^q(\tau; \xi) \) denote the function defined in (3.2) corresponding to \( \Omega_\tau \). Let \( \Delta_q(\tau) \) denote the length of the marked maximal symmetric periodic orbit of rotation number \( 1/q \) for the domain \( \Omega_\tau \), that is
\[
\Delta_q(\tau) = \max_{\xi} L^q(\tau; \xi)
\]

Lemma 3.4. For any \( q \geq 2 \), the function \( \Delta_q(\tau) : U \to \mathbb{R} \) is a locally Lipschitz function.

Proof. Let \( U' \subset U \) be any compact subset, and define
\[
K_q := \max_{\tau \in U'} \max_{\xi} \|\partial_\tau L^q(\tau; \xi)\|.
\]

\(^5\) It is easy to construct domains with 1-parameter families of such orbits.
Let us fix $\tau \in U'$, and let $\bar{\xi}$ realize the maximum of $L^q(\tau; \xi)$, that is $L^q(\tau, \bar{\xi}) = \Delta_q(\tau)$. Then $L^q(\tau'; \bar{\xi}) \leq \Delta_q(\tau')$, for any $\tau' \in U'$, hence

$$\Delta_q(\tau) - \Delta_q(\tau') \leq L^q(\tau; \bar{\xi}) - L^q(\tau'; \bar{\xi}) \leq K_q|\tau - \tau'|$$

Exchanging $\tau$ and $\tau'$ in the above inequality, we can thus conclude that:

$$|\Delta_q(\tau) - \Delta_q(\tau')| \leq K_q|\tau - \tau'|.$$  \hfill \qed

Given a continuous function $f : U \subset \mathbb{R} \rightarrow \mathbb{R}$, one can define its upper (resp. lower) differential $D^+ f(\tau)$ (resp. $D^- f(\tau)$), which is characterized as follows: $p \in D^+ f(\tau)$ (resp. $p \in D^- f(\tau)$) if and only if there exists a function $\psi \in C^1(U, \mathbb{R})$, such that $\psi'(\tau) = p$ and $\psi \geq f$ (resp. $\psi \leq f$) in a neighborhood of $\tau$, with equality at $\tau$. Both $D^+ f(\tau)$ and $D^- f(\tau)$ are convex subset of $\mathbb{R}$; they are both non empty if and only if $D^+ f(\tau) = D^- f(\tau) = \{f'(\tau)\}$.

Lemma 3.5. If $\bar{\xi}$ is a point realizing the maximum $\Delta_q(\tau) = \max_\xi L^q(\tau; \xi)$, then we have $\partial_\tau L^q(\tau; \bar{\xi}) \in D^- \Delta_q(\tau)$.

Proof. Since $\Delta_q(\cdot)$ is the maximum of $L^q(\cdot, \xi)$, if $\bar{\xi}$ realizes the maximum at $\tau$ we have $L^q(\tau', \bar{\xi}) \leq \Delta_q(\tau')$ for any $\tau' \in U$. Noticing that $L^q(\cdot; \bar{\xi})$ is a $C^1$ function, by the characterization of lower differential, we conclude that $p = \partial_\tau L^q(\tau; \bar{\xi}) \in D^- \Delta_q(\tau)$, $\square$

Remark 3.6. Indeed, we could show that $\Delta_q(t)$ is a semi-convex function and

$$D^- \Delta_q(t) = \text{co}\{\partial_\tau L^q(t; \bar{\xi})|L^q(t; \bar{\xi}) = \Delta_q(t)\}$$

Lemma 3.7. If $(\Omega_\tau)_{\tau \in U}$ is an isospectral family, then for any $q \geq 2$ and $\tau \in U$ we have $\Delta_q(\tau) = \Delta_q(0)$.

Proof. Assume that for some $\tau$ and $q \geq 2$ we have $\Delta_q(\tau) \neq \Delta_q(0)$; then since $\Delta_q(\tau)$ is continuous, $\Delta_q([0, \tau])$ contains an open set. By definition, for any $\tau$, $\Delta_q(\tau) \in \mathcal{L}(\Omega_\tau)$, but since the family is isospectral, we gather that $\Delta_q(\tau) \in \mathcal{L}(\Omega_0)$, hence $\Delta_q([0, \tau]) \subset \mathcal{L}(\Omega_0)$. Hence $\mathcal{L}(\Omega_0)$ contains an open set, but this is in contradiction with Lemma 3.1. $\square$

Let $\Omega \in \mathcal{S}^*$ and $S^\prime(\Omega) = (s^k_q; \varphi^k_q)_{k=0}^{q-1}$ be a marked symmetric maximal periodic orbit of rotation number $1/q$; then we can define the functional $\ell_{S^\prime}$ which is defined as follows: for any continuous function $\nu : \mathbb{T}^1 \rightarrow \mathbb{R}$ we let

$$\ell_{S^\prime}(\nu) := \sum_{k=0}^{q-1} \nu(s^k_q) \sin \varphi^k_q$$

(3.3)
Remark 3.8. These functionals can, of course, be defined for any periodic orbit (rather than only for marked symmetric maximal orbits). Since we will not use non-symmetric orbits for the proof, we find simpler to use the above definition.

**Proposition 3.9.** Let \((\Omega_{\tau})_{\tau \in U}\) be an isospectral family: then for any \(\tau \in U\), \(q > 1\) and \(S^q\) a maximal marked periodic orbit for \(\Omega_{\tau}\), we have
\[
\ell_{S^q}(n(\tau, \cdot)) = 0.
\]

**Proof.** Since the family \((\Omega_{\tau})_{\tau \in U}\) is isospectral, by Lemma 3.7 \(\Delta_q(\tau) \equiv \Delta_q(0)\) for any \(\tau \in U\); we thus conclude that \(\Delta_q\) is indeed differentiable and therefore, by Lemma 3.5, for any \(\tau\) fixed, let \(\xi\) realize the maximum, i.e. \(\Delta_q(\tau) = \max_{\xi} L^q(\tau; \xi) = L^q(\tau, \xi)\), we have
\[
\partial_{\tau} L^q(\tau; \xi) \in D^{-\Delta_q}(\tau) = \{0\}.
\]
In particular for \(\bar{S}^q = (\bar{s}^k, \bar{\varphi}_q)_{k=0}^{q-1}\), since \(L^q(\tau; \xi) = \sum_{k=0}^{q-1} L^q(\bar{s}^k, \bar{s}^k_q)\) and observing that
\[
\partial_{\tau} L^q(s, s') = \partial_{\tau} \|\gamma_{\tau}(s) - \gamma_{\tau}(s')\|
= \frac{\gamma_{\tau}(s) - \gamma_{\tau}(s')}{\|\gamma_{\tau}(s) - \gamma_{\tau}(s')\|} \cdot [\partial_{\tau} \gamma_{\tau}(s) - \partial_{\tau} \gamma_{\tau}(s')]
\]
we get
\[
0 = \partial_{\tau} L^q(\tau; \xi) = \partial_{\tau} \sum_{k=0}^{q-1} L^q(\bar{s}^k_q, \bar{s}^k_{q+1})
= \sum_{k=0}^{q-1} \left[ \frac{\gamma_{\tau}(\bar{s}^k_q) - \gamma_{\tau}(\bar{s}^k_{q+1})}{\|\gamma_{\tau}(\bar{s}^k_q) - \gamma_{\tau}(\bar{s}^k_{q+1})\|} - \frac{\gamma_{\tau}(\bar{s}^k_{q+1}) - \gamma_{\tau}(\bar{s}^k_q)}{\|\gamma_{\tau}(\bar{s}^k_{q+1}) - \gamma_{\tau}(\bar{s}^k_q)\|} \right] \cdot \partial_{\tau} \gamma_{\tau}(\bar{s}^k_q)
= \sum_{k=0}^{q-1} 2 \sin \varphi^k_q N_{\tau}(\bar{s}^k_q) \cdot \partial_{\tau} \gamma_{\tau}(\bar{s}^k_q)
= 2 \sum_{k=0}^{q-1} n_{\tau}(\bar{s}^k_q) \sin \varphi^k_q,
\]
which concludes the proof. \(\square\)

Remark 3.10. Let \(S = \{(\tau, \partial_{\tau} L^q(\tau; \xi))|\partial_{\tau} L^q(\tau, \xi) = 0\}\), then \(S \subset T^*U\) is a Lagrangian submanifold. Let \(\pi : T^*U \rightarrow U\) denote the natural projection, then \(L^q(t; \cdot)\) is a Morse function (critical points are non-degenerate) if and only if \(\tau\) is a regular value of the map \(\pi|_S\). The set \(U_1\) of such values is an open set, and by Sard’s theorem, it has full measure. Furthermore, the set \(U_0 \subset U_1\) of \(\tau\) such that \(L^q(\tau; \cdot)\) is an excellent Morse function (Morse function whose critical points
have pairwise distinguished critical values) is also an open subset of full measure. For any $\tau_0 \in X_0$, the critical points of $L^q(\tau; \cdot)$ depends smoothly on $\tau$ within a sufficiently small neighbourhood of $\tau_0$. Hence we have

$$\frac{d}{d\tau} \Delta_q(\tau) = \ell_{S\ell}(n_\tau), \quad \tau \in U_0$$

for an arbitrary $C^2$ deformation $\gamma_\tau$ (not necessarily isospectral).

Given a symmetric domain $\Omega \in S^r$ and having fixed a choice of marked maximal symmetric periodic orbits $\{S^q\}_{q \geq 2}$, we will use the shorthand notation $\ell_q = \ell_{S^q}$. We now proceed to define two additional functionals, which we denote with $\ell_0$ and $\ell_1$: $\ell_0(\nu)$ provides the infinitesimal change in the perimeter of $\Omega$ due to an orthogonal deformation generated by $\nu$.

It is not difficult to find an explicit formula for $\ell_0$: let $\Gamma(s)$ be an arc-length parametrization of $\partial \Omega$, and let $\delta \Gamma(s)$ be an infinitesimal deformation; then we have:

$$l_{\partial \Omega} = \int_0^{l_{\partial \Omega}} \|\Gamma'(s)\| ds.$$  

Hence, computing the variation and integrating by parts:

$$\delta l_{\partial \Omega} = \int_0^{l_{\partial \Omega}} \Gamma'(s) \cdot \delta \Gamma'(s) ds = -\int_0^{l_{\partial \Omega}} \Gamma''(s) \cdot \delta \Gamma(s)$$

$$= \int_0^{l_{\partial \Omega}} \frac{N(s) \cdot \delta \Gamma(s)}{\rho(s)} ds.$$

We conclude that, if $s$ is the arc-length parametrization of $\partial \Omega$, then $\ell_0(\nu)$ can be obtained by the formula:

$$\ell_0(\nu) = \int_0^{l_{\partial \Omega}} \frac{\nu(s)}{\rho(s)} ds.$$  

Remark 3.11. Observe that if $(\Omega_\tau)_{\tau \in U}$ is an isospectral family, we have that, for any $\Omega_\tau$, the corresponding functional $\ell_0(n_\tau) = 0$. This follows by a continuity argument analogous to the one which holds for periodic orbits and Lemma 3.1.

We now proceed to define $\ell_1(\nu)$ as the evaluation of the perturbation $\nu$ at the marked point $s = 0$, that is we simply let

$$\ell_1(\nu) = \nu(0).$$

Observe that if $\nu$ preserves our normalization condition, the marked point is fixed at the origin and, therefore, $\ell_1(\nu) = 0$. 


Let us now define the space of $C^r$-smooth even functions

$$C^r_{sym} = \{ \nu \in C^r(T^1) \text{ s.t. } \nu(s) = \nu(-s) \}.$$  

We then define the linearized isospectral operator $\mathcal{T} : C^r_{sym} \to \mathbb{R}^N$:

$$\mathcal{T}\nu = (\ell_0(\nu), \ell_1(\nu), \ldots, \ell_q(\nu), \ldots)$$

In fact, $\mathcal{T}$ has range in $\ell^\infty$, by definition of the functionals $\ell_q$, since by [2, Lemma 8], $\sin \varphi^k_q \leq C/q$ for some $C$ uniform in $q$.

Our Main Theorem then follows from

**Theorem 3.12.** If $r$ is sufficiently large and $\Omega \in \mathcal{S}^r$ is sufficiently close to a circle, the operator $\mathcal{T} : C^r_{sym} \to \ell^\infty$ is injective.

**Proof of Theorem 2.13.** Assume that any element of the family $(\Omega_\tau)_{\tau \in U}$ is sufficiently close to the circle in the sense of Theorem 3.12. Suppose by contradiction that for some $\tau \in U$, the normal deformation $n(\tau, \cdot)$ is not zero. Then, since $\mathcal{T}$ is injective, we gather that there exists $q$ so that $\ell_q(n(\tau, \cdot)) \neq 0$; this contradicts Proposition 3.9. □

The rest of this paper is devoted to the proof of Theorem 3.12.

4. A CONVENIENT COORDINATE SYSTEM

In order to investigate the properties of the functionals $\ell_q$’s it is convenient to introduce a change of variables, which conjugates the dynamics to a simpler normal form. Such coordinates are of course closely related to the Lazutkin coordinates (see [12]) and will be obtained via an approximation scheme.

**Lemma 4.1.** If $\delta$ is sufficiently small and $r$ sufficiently large, for any $\Omega \in \mathcal{D}^r$ $\delta$-close to a circle:

1. there exist $C^{r-7}$ coordinates $(\bar{x}, \bar{y})$ which conjugate the billiard map to $(\bar{x}, \bar{y}) \mapsto (\bar{x}^*, \bar{y}^*)$, where
   $$\bar{x}^* = \bar{x} + \bar{y} + O(\delta y^6)$$
   $$\bar{y}^* = \bar{y} + O(\delta y^6)$$

   where $O(\xi)$ denotes a function whose sup-norm tends to 0 at least as fast as $\xi$.

2. there exist $C^{r-5}$ functions $\tilde{\alpha}(x)$ and $\tilde{\beta}(x)$ so that $\|\tilde{\alpha}\|, \|\tilde{\beta}\| = O(\delta)$ and
   $$x = \bar{x} + \tilde{\alpha}(x)\bar{y}^2$$
   $$\varphi = \mu(x)\bar{y} + \tilde{\beta}(x)\bar{y}^3$$
where $x$ is the standard Lazutkin parameterization

\[ x(s) = C_\Omega \int_0^s \rho(\xi)^{-2/3} \, d\xi, \quad \text{where } C_\Omega = \left[ \int_{\partial \Omega} \rho(\xi)^{-2/3} \, d\xi \right]^{-1}. \]

and

\[ \mu(x) = \frac{1}{2C_\Omega \rho(x)^{1/3}}. \]

The proof of Lemma 4.1 will be given in Appendix A.2.

From the first item of Lemma 4.1 we conclude that if \((\bar{x}_q^k, \bar{y}_q^k)_{k \in \{0, \ldots, q-1\}}\) is a periodic orbit of rotation number $1/q$, we have:

\[ \bar{x}_q^k = \bar{x}_0^q + k/q + O(\delta q^{-4}) \]
\[ \bar{y}_q^k = 1/q + O(\delta q^{-5}). \]

which in turn imply that there exists $C^{r-5}$ function $\alpha$, $\beta$ and $\tilde{\beta}$ so that

\[ x_q^k = x_0^q + k/q + \frac{\alpha(x_0^q + k/q) - \alpha(x_0^q)}{q^2} + O(\delta q^{-4}) \]
\[ \varphi_q^k = \frac{\mu(x_0^q + k/q)}{q} + \frac{\tilde{\beta}(x_0^q + k/q)}{q^3} + O(\delta q^{-5}) \]

which finally yields:

\[ \sin \varphi_q^k = \sin \left( \frac{\mu(x_0^q + k/q)}{q} \right) + \frac{\beta_0(x_0^q + k/q)}{q^3} + O(\delta q^{-5}) \]
\[ = \mu(x_0^q + k/q) \frac{w_q}{q} \left( 1 + \frac{\beta(x_0^q + k/q)}{q^2} \right) + O(\delta q^{-5}) \]

where we defined for $q \geq 2$.

\[ w_q = \frac{q}{\pi} \sin \frac{\pi}{q} \]

Since we will apply the above formulas to marked maximal symmetric periodic orbits, we can set $x_0^q = 0$ and obtain the simplified expressions (observe that by normalization $\alpha(0) = 0$);

\[ x_q^k = k/q + \frac{\alpha(k/q)}{q^2} + O(\delta q^{-4}) \]
\[ \sin \varphi_q^k = \mu(k/q) \frac{w_q}{q} \left( 1 + \frac{\beta(k/q)}{q^2} \right) + O(\delta q^{-5}) \]
Lemma 4.2. If $\Omega \in S^r$, we have that $\alpha(x)$ is odd, and $\beta(x)$ is even. In particular, we have $\alpha_k = -\alpha_{-k}$ is purely imaginary, $\beta_k = \beta_{-k}$ is real:

$$\alpha(x) = \sum_{k \geq 1} 2i\alpha_k \sin 2\pi kx, \quad \beta(x) = \sum_{k \geq 0} 2\beta_k \cos 2\pi kx$$

where $\alpha_k$ and $\beta_k$ are Fourier coefficients of the functions $\alpha$ and $\beta$, respectively.

Proof. The proof is immediate from the form of (4.3). \qed

5. Functionals in Lazutkin coordinates

We now proceed to investigate the form of the functionals $\ell_q$’s in the Lazutkin coordinate obtained in the previous section. Let us introduce the Lazutkin weighting operator $W : C^0(\partial \Omega) \to C^0(\mathbb{T}^1)$ given by:

$$[Wn](x) = n(s(x))\mu(x)$$

The operator $W$ is injective; let us denote

$$\tilde{\ell}_q(u) = \ell_q(u(x(s))\mu(x(s))^{-1})$$

and correspondingly

$$\tilde{T}(u) = (\tilde{\ell}_0(u), \ldots, \tilde{\ell}_q(u), \ldots).$$

Due to the explicit formulas (4.3), the functionals $\tilde{\ell}_q$’s turn out to be simpler to study than $\ell_q$. First, observe that changing variable in the integral in the definition (3.4) (using the explicit formula (4.1)), we obtain

$$\ell_0(n) = \int_0^1 [Wn](x) \, dx,$$

hence $\tilde{\ell}_0$ is simply the functional which computes the average with respect to the Lebesgue measure. $\tilde{\ell}_1$ is the evaluation at the marked point. In order to obtain a useful expression for $\tilde{\ell}_q$ for $q \geq 2$, recall the definition

$$\ell_q(n) = \sum_{k=0}^{q-1} n(s_q^k) \sin(\varphi_q^k)$$

where $((s_q^k, \varphi_q^k))_{k \in \{0, \ldots, q-1\}}$ is a marked maximal symmetric periodic orbit. Using (4.3) we conclude:

$$\tilde{\ell}_q(u) = u_q \sum_{k=0}^{q-1} u_k \left( k/q + \frac{\alpha(k/q)}{q^2} \right) \left( 1 + \frac{\beta(k/q)}{q^2} \right) + O(\delta q^{-4})$$
Let us now define the Fourier basis $B = (e_j)_{j \geq 0}$ of even functions on the circle in Lazutkin parametrization:

$$e_j = \cos 2\pi j x \quad j \geq 0.$$ 

Let us introduce the convenient notation

$$\delta_{q|j} = \begin{cases} 
1 & \text{if } q|j \\
0 & \text{otherwise}
\end{cases}$$

**Lemma 5.1.** For any $\Omega$ there exists a functional $\tilde{\ell}_*$ so that for any $q \geq 2, j \geq 1$:

$$\tilde{\ell}_*(e_j) w_q^{-1} = \frac{\tilde{\ell}_*(e_j)}{q^2} + \left(1 + \frac{\beta_0}{q^2}\right) \delta_{q|j} +$$

$$+ \frac{1}{q^2} \sum_{s \in \mathbb{Z}\setminus\{0\}} \sum_{sq \neq j} \left(\beta_{sq-j} + 2\pi i j \alpha_{sq-j}\right) + O(\delta j^2 q^{-4})$$

where $\tilde{\ell}_*(e_j) = \beta_j - 2\pi i j \alpha_j$, and $\alpha_j, \beta_j$ are Fourier coefficients in Lemma 4.2.

**Proof.** The lemma follows from a simple computation: by plugging in $u = e_j$ in (5.1) we obtain:

$$\tilde{\ell}_*(e_j) w_q^{-1} = \frac{1}{q} \sum_{k=0}^{q-1} \cos \left(2\pi j \left(\frac{k}{q} + \frac{\alpha(k/q)}{q^2}\right)\right) \left(1 + \frac{\beta(k/q)}{q^2}\right) + O(\delta q^{-4})$$

$$= \frac{1}{q} \sum_{k=0}^{q-1} \left(\cos(2\pi j k/q) - 2\pi j \sin(2\pi j k/q) \frac{\alpha(k/q)}{q^2}\right) \left(1 + \frac{\beta(k/q)}{q^2}\right)$$

$$+ O(\delta j^2 q^{-4})$$

$$= \delta_{q|j} + \frac{1}{2q^3} \sum_{k=0}^{q-1} \sum_{p \in \mathbb{Z}} \left[2\pi i j (\exp(2\pi i j k/q) - \exp(-2\pi i j k/q)) \alpha_p \exp(2\pi i p k/q) +
+ (\exp(2\pi i j k/q) + \exp(-2\pi i j k/q)) \beta_p \exp(2\pi i p k/q) \right]$$

$$+ O(\delta j^2 q^{-4})$$

$$= \delta_{q|j} + \frac{1}{2q^2} \sum_{s \in \mathbb{Z}} \left[\beta_{sq-j} + 2\pi i j \alpha_{sq-j} + \beta_{sq+j} - 2\pi i j \alpha_{sq+j}\right] + O(\delta j^2 q^{-4})$$
and using the fact that $\alpha_p = -\alpha_{-p}$ and $\beta_p = \beta_{-p}$

$$\alpha_{qj} = \frac{1}{q^2} \sum_{s \in \mathbb{Z}} [\beta_{sq-j} + 2\pi i j \alpha_{sq-j}] + O(\delta j^2 q^{-4})$$

$$\alpha_{qj} = \left(1 + \frac{\beta_0}{q^2}\right) \alpha_{qj} + \frac{\beta_j - 2\pi i j \alpha_j}{q^2} +$$

$$+ \frac{1}{q^2} \sum_{s \in \mathbb{Z} \setminus \{0\}} (\beta_{sq-j} + 2\pi i j \alpha_{sq-j}) + O(\delta j^2 q^{-4}).$$

which proves our lemma setting $\ell_q(e_j) = \beta_j - 2\pi i j \alpha_j$. \hfill \Box

6. Proof of Theorem 3.12

First of all let us introduce the projector $P_* : C_{sym}^r \to C_{sym}^r$, where $C_{sym}^r$ is the space of real, even, zero average $C^r$-functions of $T^1$: $P_* u = u - \int u(x) dx$.

Let $L^1_{*,sym}$ be the space of $L^1$, even, zero average, functions of the circle, i.e.

$$L^1_{*,sym} = \left\{ u \in L^1(T^1) \text{ s.t. } u(x) = u(-x), \int u(x) dx = 0 \right\}$$

$$= \left\{ u \in L^1(T^1) \text{ s.t. } u(x) = \sum_{j \geq 1} \hat{u}_j e_j \right\}$$

where $\hat{u}_k$ denote its Fourier coefficients in the basis $B$.

We now proceed to define a space of admissible functions: for $3 < \gamma < 4$, define the subspace $X_{*\gamma} = \{ u \in L^1_{*,sym} \text{ s.t. } \lim_{j \to \infty} j^\gamma |\hat{u}_j| = 0 \}$; equipped with the norm:

$$\|u\|_{\gamma} = \max_{j \geq 1} j^\gamma |\hat{u}_j|.$$

The space $(X_{*\gamma}, \|\cdot\|_{\gamma})$ is a (separable) Banach space.

Remark 6.1. By definition $C_{*,sym}^{[\gamma]-1} \subset X_{*\gamma} \subset C_{*,sym}^{[\gamma]-1}$, whence the functionals

$$\tilde{\ell}_q(n) = \tilde{\ell}_q \left( \sum_{j=1}^{\infty} n_j e_j \right) = \sum_{j=1}^{\infty} n_j \tilde{\ell}_q(e_j).$$

are well-defined on $X_{*\gamma}$, since the Fourier series converges uniformly.
We will choose $\gamma > 3$. We will always assume $r > \gamma$.

Let $\ell_\infty^* = \{ b = (a_i)_{i \geq 0} \in \ell_\infty \text{ s.t. } a_0 = 0 \}$ and let us introduce the subspace

$$h_{*, \gamma} = \{ b = (a_i)_{i \geq 0} \in \ell_\infty^* \text{ s.t. } \lim_{j \to \infty} j^\gamma a_j = 0 \}$$

equipped with the norm $|b|_\gamma = \max_{j \geq 0} j^\gamma |a_j|$;

**Lemma 6.2.** There exist linearly independent vectors $b_1, b_* \in \ell_\infty \setminus h_{*, \gamma}$ so that, for any $\Omega \in S^r$ with $r$ sufficiently large, $\tilde{T} : C^r_{\text{sym}} \to \ell_\infty$ can be decomposed as follows:

$$\tilde{T} = b_1 \tilde{\ell}_0 + \left[ b_* \tilde{\ell}_* + \tilde{T}_{*, R} \right] P_*$$

where $\tilde{T}_{*, R} : X_{*, \gamma} \to h_{*, \gamma}$ is an invertible operator.

**Proof of Theorem 3.12.** By assumptions, the vectors $b_1, b_*$ and $\tilde{T}_{*, R} P_* u$ are linearly independent for any $u \in C^r_{\text{sym}}$. Hence, if $u \in \ker \tilde{T}$, then necessarily $\tilde{\ell}_0(u) = \tilde{\ell}_*(u) = 0$ and $\tilde{T}_{*, R} P_* (u) = 0$. Now, by definition, if $\tilde{\ell}_0(u) = 0$, then $u = P_* u \in C^r_{*, \text{sym}}$. Since $\tilde{T}_{*, R}$ is injective and $\tilde{T}_{*, R} u = 0$ we thus conclude that $u = 0$. \hfill \Box

**Proof of Lemma 6.2.** Let us first decompose

$$\tilde{T} = \tilde{T}((1 - P_*) + P_*) = \tilde{T}(1 - P_*) + \tilde{T} P_*$$

where $\tilde{T}_*$ is the restriction of $\tilde{T}$ on $C^r_{*, \text{sym}}$. Observe that, by definition $(1 - P_*) u$ is the constant function equal to $\tilde{\ell}_0(u)$; we can thus set

$$b_1 := \tilde{T}(1) = (1, 1, w_2 + O(\delta), \ldots, w_q + O(\delta q^{-2}), \ldots) \in \ell_\infty.$$

We thus conclude that $\tilde{T} = b_1 \tilde{\ell}_0 + \tilde{T}_* P_*$. Observe moreover that by Lemma 5.1 we can write $\tilde{T}_* = b_* \tilde{\ell}_* + \tilde{T}_{*, R}$, where

$$b_* = (0, 0, w_2/4, \ldots, w_q/q^2, \ldots)$$

and recall that

$$\tilde{\ell}_*(u) = \tilde{\ell}_*(-\sum_{j \geq 1} \hat{u}_j e_j) = \sum_{j \geq 1} \hat{u}_j (\beta_j - 2\pi j \alpha_j)$$

$\tilde{\ell}_*(u)$ seems to define on the projection space $X_{*, \gamma}$.

Clearly, $b_*$ and $b_1$ are linearly independent and do not belong to $h_{*, \gamma}$ since $\gamma \geq 2$. On the other hand, let $\tilde{T}_{qj}$ denote the matrix representation of $\tilde{T}_{*, R}$ in the canonical bases.
For $q = 1$, we have by definition $\tilde{T}_{1j} = \tilde{\ell}_1(e_j) = 1$; on the other hand, for $q \geq 2$, Lemma 5.1 yields the expression:

$$\tilde{T}_{qj} = w_q \left(1 + \frac{\beta_0}{q^2}\right) \left[\delta_{qj} + \frac{1}{q^2} \sum_{s \in \mathbb{Z} \setminus \{0\}, sq \neq j} (\beta_{sq-j} + 2\pi ij\alpha_{sq-j}) + O(\delta^2q^{-4})\right]$$

$$= w_q \left(1 + \frac{\beta_0}{q^2}\right) [\Delta_{qj} + R_{qj}]$$

where $\Delta_{qj} = \delta_{qj}$ and $R_{qj}$ is the remainder. Clearly, it suffices to show that $[\Delta_{qj} + R_{qj}]$ is invertible provided that $\delta$ is sufficiently small. We will in fact show that

$$\|\Delta + R - \text{Id}\|_\gamma < 1.$$  \hspace{1cm} (6.1)

where $\| \cdot \|_\gamma$ is the operator norm from $(X_{*, \gamma}, \| \cdot \|_\gamma)$ to $(h_{*, \gamma}, | \cdot |_\gamma)$. The first step is to establish the following

**Lemma 6.3.** The operator $\Delta : X_{*, \gamma} \to h_{*, \gamma}$, identified by the matrix $\Delta_{qj}$ is such that $\|\Delta - \text{Id}\|_\gamma < 2/3$; in particular $\Delta$ has bounded inverse.

**Proof.** By definition the norm $\|\Delta - \text{Id}\|_\gamma$ is given by:

$$\|\Delta - \text{Id}\|_\gamma = \sup_q q^\gamma \sum_{j > 0} (j^{-\gamma}\delta_{qj} - \delta_{qj}) = \sup_q q^\gamma \left[q^\gamma \sum_{j > 0} j^{-\gamma}\delta_{qj}\right] - 1$$

$$= \sup_q q^\gamma \sum_{s > 0} (sq)^{-\gamma} - 1 \leq \sum_{s > 0} s^{-2} - 1 = \frac{\pi^2}{6} - 1$$

since $\gamma > 2$. \hfill \Box

In order to conclude the proof of (6.1) it thus suffices to show that by taking $\delta$ sufficiently small, we can make $\|R_{qj}\|_\gamma < 1/3$.

First, we check the term $O(\delta j^2 q^{-4})$; we obtain

$$O(\delta) \sup_q \sum_{j > 0} O(2^{-\gamma} q^{-4}).$$

We thus need $\gamma > 3$ for the sum on $j$ to converge, and $\gamma < 4$ for the sequence to converge to 0 as $q \to \infty$. We conclude that this term can be made as small as needed by choosing $\delta$ sufficiently small.

\[\delta_{qj}\] is the usual Kronecker delta notation.
Next, we deal with the sum: observe that, since $\alpha$ and $\beta$ are smooth

$$j^{-\gamma} \left| \sum_{s \in \mathbb{Z} \setminus \{0\}} (\beta_{sq - j} + 2\pi ij \alpha_{sq - j}) \right| < O(\delta) q^{-2} \sum_{s \in \mathbb{Z} \setminus \{0\}} \frac{1}{|sq - j|^r j^{-\gamma}}$$

Where we can choose $r'$ as large as needed by choosing $r$ large. Let us now estimate once again the sum over $j$:

$$q^{-2} \sum_{s \geq 0} \frac{1}{|sq - j|^r j^{-\gamma}} = q^{-2} \sum_{s \geq 0} \left( \sum_{0 < j < |sq|^+} + \sum_{j > |sq|^+} \right) \frac{1}{|sq - j|^r j^{-\gamma}} = I + II$$

where $|sq|^+ = \max\{0, sq\}$. Then

$$II \leq q^{-2} \sum_{s \geq 0} \frac{1}{|sq|^+} \sum_{j > |sq|^+} \frac{1}{|sq - j|^r} = \frac{C}{q} \sum_{s \geq 0} |s|^{-\gamma} < \frac{C}{q}$$

then we write:

$$I = q^{-2} \sum_{s \geq 0} \left( \sum_{0 < j < sq/2} + \sum_{sq/2 \leq j < sq} \right) \frac{1}{|sq - j|^r j^{-\gamma}} = I' + I''$$

Then

$$I' \leq q^{-2} \sum_{s > 0} \sum_{0 < j < sq/2} \frac{2^{1 + r'}}{|sq|^r j^{-\gamma}} < \frac{C}{q^{r - \gamma}} \sum_{s > 0} \sum_{j > 0} \frac{1}{s^{1 + \gamma}} < \frac{C}{q^{r - \gamma}}$$

$$I'' \leq q^{-2} \sum_{s > 0} \sum_{sq/2 < j < sq} \frac{1}{|sq - j|^r |sq|^{-\gamma}} < \frac{C}{q} \sum_{s > 0} \sum_{p > 0} \frac{1}{p^r s^{1 - \gamma}} < \frac{C}{q}$$

which allows to conclude provided $\gamma < r'$. \qed

**APPENDIX A. LAZUTKIN COORDINATES**

**A.1. An abstract setting.** Let $F : S^1 \times S^1 \to S^1 \times S^1$ be a monotone orientation preserving twist $C^s$ diffeomorphism with an involution given by $I : (x, y) \mapsto (x, -y)$, (that is $I \circ F^{-1} = F \circ I$). Observe in particular that $F$ fixes the circle $y = 0$, that is $F(x, 0) = (x, 0)$. Assume moreover that the lift of $F$ to $\mathbb{R}^2$ satisfies the condition $F(x, y + 2\pi) = (x + 2\pi, y + 2\pi)$.

Then let us introduce a convenient definition: for $N \geq 0$ consider a solution $\mathcal{F} : C^s(S^1 \times S^1, S^1)$ of the following equation:

$$\mathcal{F} \circ F - \mathcal{F} = \mathcal{F} - \mathcal{F} \circ F^{-1} + O(y^N).$$
Using the properties of $I$ we can rewrite (A.1) as

$$L \circ F + L \circ I \circ F \circ I - L = O(\varphi^N).$$

Observe now that if $L$ solves the above equation, then $L \circ I$ is also a solution. By linearity of the equation, we further conclude that $L$ can always assumed to be invariant under $I$, that is, an even function of $y$.

Given a function $L$, we can define its even and odd part (with respect to the second coordinate) as:

$$[L]^{\text{even}} = \frac{1}{2} [L + L \circ I] \quad [L]^{\text{odd}} = \frac{1}{2} [L - L \circ I]$$

Such function will, thus, satisfy the following symmetrized coboundary equation:

$$[L \circ F]^{\text{even}} - L = O(y^{2K})$$

where we set $N = 2K$ since the left hand side is an even function of $y$.

An even function $L \in C^s(S^1 \times S^1, S^1)$ is called a Lazutkin function of order $2K$ if

- for any fixed $y \in S^1$ the map $L(\cdot, y) : S^1 \to S^1$ is an orientation-preserving diffeomorphism.
- $L$ satisfies (A.2).

Let us now define $Y_{\varpi}(x, y)$ as the following real function, odd in $y$:

$$Y_{\varpi} = [L \circ F]^{\text{odd}}$$

Observe that since $F$ is a twist map, $Y_{\varpi}(x, \cdot)$ is strictly increasing for any fixed $x$. Moreover, by our assumptions on $F$ we gather that $Y_{\varpi}(x, 2\pi) = Y_{\varpi}(x, 0) + 2\pi$. We conclude that, for any $x \in S^1$, the function $Y_{\varpi}(x, \cdot)$ on a diffeomorphism $y_{\varpi}(x, \cdot) : S^1 \to S^1$ in other words, setting $x_{\varpi}(x, y) = \varpi(x, y)$ we have that $(x, y) \mapsto (x_{\varpi}, y_{\varpi})$ is a non-degenerate change of coordinates. It is simple to see that in these coordinates the map $F$ reads

$$F_{\varpi} : \begin{cases} x_{\varpi} \mapsto x_{\varpi} + y_{\varpi} + r_0(x_{\varpi}, y_{\varpi})y_{\varpi}^{2K} \\ y_{\varpi} \mapsto y_{\varpi} + r_1(x_{\varpi}, y_{\varpi})y_{\varpi}^{2K} \end{cases}$$

where $r_0, r_1 \in C^s(S^1 \times S^1)$ and, moreover:

$$r_0(x_{\varpi}, y_{\varpi})y_{\varpi}^{2K} = [L \circ F(x, y)]^{\text{even}} - L(x, y)$$

and $(x, y)$ are seen as function of $(x_{\varpi}, y_{\varpi})$. Moreover, since $y_{\varpi}$ is an odd function of $y$, we gather that the involution has the same form $I : (x_{\varpi}, y_{\varpi}) \mapsto (x_{\varpi}, -y_{\varpi})$ in such coordinates. In particular, $r_0$ is an even function of $y$. We call $(x_{\varpi}, y_{\varpi})$ Lazutkin variables of order $2K$. 


Remark A.1. Observe that since $F$ fixes the circle $y = 0$, $\Pi(x, y) = x$ is a Lazutkin function\textsuperscript{7} of order 2, provided $F \in C^3$. In particular, replacing $y$ with $[F(x, y) - x]^{\text{odd}} = \partial_y x^*(x, 0)y + O(y^3)$ we can always assume that $F$ is given the canonical form $F$:

\[
F : \begin{cases} 
  x \mapsto x + y + r_0(x, y)y^2 \\
  y \mapsto y + r_1(x, y)y^2.
\end{cases}
\]  

(A.4)

The following lemma constitutes the main result of this section. It states that provided that $r_0$ and $r_1$ are sufficiently small and sufficiently smooth, then we can always find Lazutkin coordinates of higher order.

Lemma A.2. Assume that for some $K > 0$ and $s \geq 3$, there exist $C^s$ functions $r_0, r_1$ so that:

\[
F : \begin{cases} 
  x \mapsto x + y + r_0(x, y)y^{2K} \\
  y \mapsto y + r_1(x, y)y^{2K}
\end{cases}
\]

then if $\|r_0\|_{C^s}$ is sufficiently small\textsuperscript{8}, there exists a $C^s$ change of coordinates $(x, y) \mapsto (\bar{x}, \bar{y})$ and $C^{s-2}$ functions $\bar{r}_0, \bar{r}_1$ so that

\[
\bar{F} : \begin{cases} 
  \bar{x} \mapsto \bar{x} + \bar{y} + \bar{r}_0(\bar{x}, \bar{y})\bar{y}^{2K+2} \\
  \bar{y} \mapsto \bar{y} + \bar{r}_1(\bar{x}, \bar{y})\bar{y}^{2K+2}
\end{cases}
\]

Moreover for $i \in \{0, 1\}$, we have $\|\bar{r}_i\|_{C^{s-2}} \leq C(\|r_0\|_{C^s} + \|r_1\|_{C^s})$.

Proof. The key to the proof is to find suitable Lazutkin functions; to simplify the exposition it is convenient to treat separately the case $K = 1$ and $K > 1$.

Let us first consider the case $K = 1$ (which corresponds to (A.4)); then we look for a Lazutkin function $\Pi$ of order 4. We will make the ansatz that $\Pi(x, y) = \pi(x)$, where $\pi$ solves the differential equation

\[
2\pi'(x)r_0(x, 0) + \pi''(x) = 0
\]

(A.5)

with boundary conditions $\pi(0) = 0$ and $\pi(2\pi) = 2\pi$. Let us prove that $\Pi$ is a Lazutkin function: first of all, ODE considerations imply that $\pi' > 0$, which, together with the boundary conditions, implies that $x \mapsto \Pi(x, y)$ is a circle diffeomorphism for any $y$. We thus need to show that $\Pi$ satisfies (A.2). First of all observe that we can write

\[
r_0(x, y) = r_0(x, 0) + \tilde{r}_0(x, y)y^2
\]

\textsuperscript{7} As a matter of fact, any diffeomorphism independent of $\varphi$ would give a Lazutkin function of order 2.

\textsuperscript{8} This assumption is actually not needed if $K = 1$.
where $\tilde{r}_0$ is a $C^{s-2}$ function. For simplicity, let $F(x, y) = (x^*, y^*)$; then we can write
\[
\pi(x^*) = \pi(x) + \pi'(x)[y + r_0(x, 0)y^2] + \frac{\pi''(x)}{2}y^2 + \tilde{\pi}(x, y)y^3
\]
\[
= \pi(x) + \pi'(x)y + \left(\pi'(x)r_0(x, 0) + \frac{\pi''(x)}{2}\right)y^2 + \tilde{\pi}(x, y)y^3
\]
and by (A.5) we conclude
\[
= \pi(x) + \pi'(x)y + \tilde{\pi}(x, y)y^3
\]
where $\tilde{\pi}$ is $C^{s-2}$. Hence (A.2) is satisfied with a $C^{s-2}$ remainder that we denote with $\bar{r}_0$. The estimates on the norms are immediate. The case $K > 1$ is similar, but simpler: this time the ansatz is $L(x, y) = x + \pi(x)y^{2(K-1)}$, where we assume that
\[
\pi''(x) = -2r_0(x, 0)
\]
with boundary conditions $\pi(0) = \pi(2\pi) = 0$. Then, if $\|r_0\|_{C^0}$ is sufficiently small $\partial_x L(x, y) = 1 + \pi'(x)y^2 > 0$, thus $L(\cdot, y)$ is a circle diffeomorphism. Moreover, again we can write:
\[
L(x^*, y^*) = x + y + r_0(x, 0)y^{2K} + \pi(x)y^{2K-1}
\]
\[
+ \pi'(x)y^{2K-1} + \frac{\pi''(x)}{2}y^{2K} + \tilde{\pi}(x, y)y^{2K+1}
\]
\[
= x + \pi(x)y^{2K-1} + y
\]
\[
+ \pi'(x)y^{2K-1} + \left(\pi'(x)r_0(x, 0) + \frac{\pi''(x)}{2}\right)y^{2K} + \tilde{\pi}(x, y)y^{2K+1}
\]
and again using our ansatz
\[
= L(x, y) + y + \pi'(x)y^{2K-1} + \tilde{\pi}(x, y)y^{2K+1}.
\]
We thus conclude, by taking the even part, that $L(x, y)$ is a Lazutkin function of order $2K + 2$ with $C^{s-2}$ remainder which we call $\bar{r}_0(x, y)$. Once again, the estimates on the norms are immediate.

A.2. Billiards: the proof of Lemma 4.1. We now proceed to apply the result of the previous section to the case of billiard dynamics. Let us fix a domain $\Omega$ and denote with $(s^+(s, \varphi), \varphi^+(s, \varphi))$ (resp. $(s^-(s, \varphi), \varphi^-(s, \varphi))$) the coordinates of the image (resp. preimage) of $(s, \varphi)$ under the billiard map $f = f_\Omega$, where recall that $s$ is the arc-length parameter on $\partial \Omega$ and $\varphi \in [0, \pi]$ is the angle that the outgoing trajectory forms with the positively oriented tangent at $s$. It follows
from the definition that the map \( I(s, \varphi) \mapsto (s, \pi - \varphi) \) is an involution for the billiard dynamics, that is \( I \circ f^{-1} = f \circ I \), or more explicitly

\[
(s^-(s, \varphi), \pi - \varphi^-(s, \varphi)) = s^+(s, \pi - \varphi), \varphi^+(s, \pi - \varphi).
\]

For convenience, we will consider the angle \( \varphi \) to be defined on \( S^1 \), rather than on \([0, \pi]\), via the identification \( \varphi \sim \varphi + \pi \). Then, by the properties of the involution we have:

\[
\begin{align*}
    s^-(s, \varphi) &= s^+(s, \pi - \varphi) = s^+(s, -\varphi) \\
    \varphi^-(s, \varphi) &= \pi - \varphi^+(s, \pi - \varphi) = -\varphi^+(s, -\varphi)
\end{align*}
\]

For symmetry of notation, let us now denote

\[
    s^* = s^+ \quad \text{and} \quad \varphi^* = \varphi^+.
\]

It follows from simple geometrical observations that \( s^* \) and \( \varphi^* \) are both smooth at \( \varphi = 0 \).

As described in Remark A.1, we can put the billiard map in the canonical form (A.4) by choosing \( x = s \), which gives coordinate reads:

\[
y(s, \varphi) = 2\rho(s) \varphi + O(\varphi^2)
\]

**Remark A.3.** Notice that if \( \partial \Omega \) is a circle, the \( \rho(s) \) is constant equal to the radius of the circle and the billiard map reduces to:

\[
f : \begin{cases}
x \mapsto x + y \\
y \mapsto y
\end{cases}
\]

By following the procedure described in the proof of Lemma A.2, we can obtain Lazutkin coordinates of order 4 by means of the change of variables

\[
\bar{y}(x) = \pi(x) = C_\Omega \int_0^x \rho(\xi)^{-2/3} \, d\xi, \quad \text{where} \quad C_\Omega = \left( \int_{\partial \Omega} \rho(\xi)^{-2/3} d\xi \right)^{-1}.
\]

and expressing \( x^* = x + \sum_k b_k y^k \), we obtain the explicit formula

\[
\bar{y} = x'b_1 y + \left[ x'b_3 + x'b_1 b_2 + \frac{\pi^{m}}{6} b_1^3 \right] y^3 = \\
= C_\Omega \rho^{-2/3} \left[ y + \frac{2\rho^2 - 3\rho \ddot{\rho}}{108\rho^2} y^3 \right] = 2C_\Omega \rho^{1/3} \left[ \varphi + \frac{2\rho^2 - 3\rho \ddot{\rho}}{27\rho^3} \varphi^3 \right]
\]

Notice that, while the \( \bar{x} \) coordinate is the same as the one used by Lazutkin (see [12]), the \( \bar{y} \) coordinate is not (they differ at order \( \varphi^3 \)): in fact in these coordinates the dynamics is conjugated to (A.3) with \( K = 2 \) with remainder term \( \|r_0\|_C^k \leq C\|\rho\|_{C^{2+k}}^3 \). Lemma 4.1 then
follows by applying Lemma A.2 to find Lazutkin coordinates of order 6.

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