PROPER AFFINE ACTIONS AND GEODESIC FLOWS OF HYPERBOLIC SURFACES

W. GOLDMAN, F. LABOURIE, AND G. MARGULIS

Abstract. Let $\Gamma_0 \subset \mathsf{O}(2,1)$ be a Schottky group, and let $\Sigma = \mathsf{H}^2/\Gamma_0$ be the corresponding hyperbolic surface. Let $\mathcal{C}(\Sigma)$ denote the space of geodesic currents on Σ. The cohomology group $H^1(\Gamma_0, \mathsf{V})$ parametrizes equivalence classes of affine deformations Γ_{u} of Γ_0 acting on an irreducible representation V of $\mathsf{O}(2,1)$. We define a continuous biaffine map

$$\mathcal{C}(\Sigma) \times H^1(\Gamma_0, \mathsf{V}) \xrightarrow{\Psi} \mathbb{R}$$

which is linear on the vector space $H^1(\Gamma_0, \mathsf{V})$. An affine deformation Γ_{u} acts properly if and only if $\Psi(\mu, [\mathsf{u}]) \neq 0$ for all $\mu \in \mathcal{C}(\Sigma)$. Consequently the set of proper affine actions whose linear part is a Schottky group identifies with a bundle of open convex cones in $H^1(\Gamma_0, \mathsf{V})$ over the Teichmüller space of Σ.

Contents

Date: March 2, 2006.

1991 *Mathematics Subject Classification.* 57M05 (Low-dimensional topology), 20H10 (Fuchsian groups and their generalizations).

Key words and phrases. proper action, hyperbolic surface, Schottky group, geodesic flow, geodesic current, flat vector bundle, flat affine bundle, homogeneous bundle, flat Lorentzian metric, affine connection.

Goldman gratefully acknowledge partial support from National Science Foundation grant DMS-0103889. Margulis gratefully acknowledge partial support from National Science Foundation grant DMS-0244406. Labourie thanks l'Institut Universitaire de France.

Introduction

It is well known that every discrete group of Euclidean isometries of $\mathbb{R}^n$ contains a free abelian subgroup of finite index. In [31], Milnor asked if every discrete subgroup of *affine transformations* of $\mathbb{R}^n$ must contain a polycyclic subgroup of finite index. Furthermore he showed that this question is equivalent to whether a discrete subgroup of $\mathsf{Aff}(\mathbb{R}^n)$ which acts properly must be *amenable.* By Tits [36], this is equivalent to the existence of a proper affine action of a nonabelian free group. Margulis subsequently showed [28, 29] that proper affine actions of nonabelian free groups do indeed exist. The present paper completes the classification of proper affine actions on $\mathbb{R}^3$ for a large class of discrete groups.

If $\Gamma \subset \mathsf{Aff}(\mathbb{R}^n)$ is a discrete subgroup which acts properly on $\mathbb{R}^n$, then a subgroup of finite index will act freely. In this case the quotient $\mathbb{R}^n/\Gamma$ is a *complete affine manifold* M with fundamental group $\pi_1(M) \cong \Gamma$. When $n = 3$, Fried-Goldman [19] classified all the amenable such Γ (including the case when M is compact). Furthermore the linear holonomy homomorphism

$$\pi_1(M) \xrightarrow{\mathbb{L}} \mathsf{GL}(\mathbb{R}^3)$$

embeds $\pi_1(M)$ onto a discrete subgroup $\Gamma_0 \subset \mathsf{GL}(\mathbb{R}^3)$, which is conjugate to a subgroup of $\mathsf{O}(2,1)^0$ (Fried-Goldman [19]). Mess [30] proved the hyperbolic surface

$$\Sigma = \Gamma_0 \backslash \mathsf{O}(2,1)^0$$

is noncompact. Goldman-Margulis [22] and Labourie [27] gave alternate proofs of Mess's theorem, using ideas which evolved to the present work. Labourie's result applies to affine deformations of Fuchsian actions in all dimensions.

Thus the classification of proper 3-dimensional affine actions reduces to *affine deformations* of free discrete subgroups $\Gamma_0 \subset \mathsf{O}(2,1)^0$. An *affine deformation* of Γ_0 is a group Γ of affine transformations whose linear part equals Γ_0, that is, a subgroup $\Gamma \subset \mathsf{Isom}^0(\mathbb{E}^{2,1})$ such that the restriction of $\mathbb{L}$ to Γ is an isomorphism $\Gamma \longrightarrow \Gamma_0$.

Equivalence classes of affine deformations of $\Gamma_0 \subset \mathsf{O}(2,1)^0$ are parametrized by the cohomology group $H^1(\Gamma_0, \mathsf{V})$, where V is the linear holonomy representation. Given a cocycle $\mathsf{u} \in Z^1(\Gamma_0, \mathsf{V})$, we denote the corresponding affine deformation by Γ_{u}. Drumm [12, 13, 15, 11] showed that Mess's necessary condition of non-compactness is sufficient: *every noncocompact discrete subgroup of* $\mathsf{O}(2,1)^0$ *admits proper affine deformations.* In particular he found an open subset of $H^1(\Gamma_0, \mathsf{V})$ parametrizing proper affine deformations [17].

This paper gives a criterion for the properness of an affine deformation Γ_{u} in terms of the parameter $[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V})$.

With no extra work, we take V to be representation of Γ_0 obtained by composition of the discrete embedding $\Gamma_0 \subset \mathsf{SL}(2,\mathbb{R})$ with any irreducible representation of $\mathsf{SL}(2,\mathbb{R})$. Such an action is called *Fuchsian* in Labourie [27]. Proper actions occur only when V has dimension $4k+3$ (see Abels [1], Abels-Margulis-Soifer [2, 3] and Labourie [27]). In those dimensions, examples of proper affine deformations of Fuchsian actions can be constructed using methods of Abels-Margulis-Soifer [2, 3].

Theorem. *Suppose that Γ_0 contains no parabolic elements. Then the equivalence classes of proper affine deformations of Γ form an open convex cone in $H^1(\Gamma_0, \mathsf{V})$.*

The main technique in this paper involves a generalization of the invariant constructed by Margulis [28, 29] and extended to higher dimensions by Labourie [27]. For an affine deformation Γ_{u}, a class function $\Gamma_0 \xrightarrow{\alpha_{\mathsf{u}}} \mathbb{R}$ exists which satisfies the following properties:

- $\alpha_{[\mathsf{u}]}(\gamma^n) = |n|\alpha_{[\mathsf{u}]}(\gamma)$ (when r is odd);
- $\alpha_{[\mathsf{u}]}(\gamma) = 0 \iff \gamma$ fixes a point in E.
- The function $\alpha_{[\mathsf{u}]}$ depends linearly on $[\mathsf{u}]$;
- The map
$$\begin{aligned} H^1(\Gamma_0, \mathsf{V}) &\longrightarrow \mathbb{R}^{\Gamma_0} \\ [u] &\longmapsto \alpha_{[u]} \end{aligned}$$
is injective ([18]).
- Suppose $\dim \mathsf{V} = 3$. If $\Gamma_{[\mathsf{u}]}$ acts properly, then $|\alpha_{[\mathsf{u}]}(\gamma)|$ is the Lorentzian length of the unique closed geodesic in $\mathsf{E}/\Gamma_{[\mathsf{u}]}$ corresponding to γ.
- (Opposite Sign Lemma) Suppose Γ acts properly. Either $\alpha_{[\mathsf{u}]}(\gamma) > 0$ for all $\gamma \in \Gamma_0$ or $\alpha_{[\mathsf{u}]}(\gamma) < 0$ for all $\gamma \in \Gamma_0$.

This paper provides a more conceptual proof of the Opposite Sign Lemma by extending Margulis's invariant to a continuous function Ψ_{u}on a connected space $\mathcal{C}(\Sigma)$ of probability measures. If
$$\alpha_{[\mathsf{u}]}(\gamma_1) < 0 < \alpha_{[\mathsf{u}]}(\gamma_2),$$
then an element $\mu \in \mathcal{C}(\Sigma)$ "between" γ_1 and γ_2 exists, satisfying $\Psi_{\mathsf{u}}(\mu) = 0$.

Associated to the linear group Ω_0 is a complete hyperbolic surface $\Sigma = \Gamma_0\backslash\mathsf{H}^2$. A *geodesic current* [6] is a Borel probability measure on the unit tangent bundle $U\Sigma$ of Σ invariant under the geodesic flow ϕ. We modify $\alpha_{[\mathsf{u}]}$ by dividing it by the length of the corresponding closed geodesic in the hyperbolic surface. This modification extends to a continuous function $\Psi_{[\mathsf{u}]}$ on the compact space $\mathcal{C}(\Sigma)$ of *geodesic currents* on Σ.

Here is a brief description of the construction, which first appeared in Labourie [27]. For brevity we only describe this for the case $\dim \mathsf{V} = 3$. Corresponding to the affine deformation $\Gamma_{[\mathsf{u}]}$ is a flat affine bundle $\mathbb{E}$ over $U\Sigma$, whose associated flat vector bundle has a parallel Lorentzian structure $\mathbb{B}$. Let ξ be the vector field on $U\Sigma$ generating the geodesic flow. For any section s of $\mathbb{E}$, its covariant derivative $\nabla_\xi(s)$ is a section of the flat vector bundle $\mathbb{V}$ associated to $\mathbb{E}$. Let ν denote the section of $\mathbb{V}$ which associates to a point $x \in U\Sigma$ the unit-spacelike vector corresponding to the geodesic associated to x. Let $\mu \in \mathcal{C}(\Sigma)$ be a geodesic current. Then $\Psi_{[\mathsf{u}]}(\mu)$ is defined by:

$$\int_{U\Sigma} \mathbb{B}\big(\nabla_\xi(s), \nu\big)\, d\mu.$$

Nontrivial elements $\gamma \in \Gamma_0$ correspond to periodic orbits c_γ for ϕ. The period of c_γ equals the length $\ell(\gamma)$ of the corresponding closed geodesic in Σ. Denote the subset of $\mathcal{C}(\Sigma)$ consisting of measures supported on periodic orbits by $\mathcal{C}_{\mathsf{per}}(\Sigma)$. Denote by μ_γ the geodesic current corresponding to c_γ. Because $\alpha(\gamma^n) = |n|\alpha(\gamma)$ and $\ell(\gamma^n) = |n|\ell(\gamma)$, the ratio $\alpha(\gamma)/\ell(\gamma)$ depends only on μ_γ.

Theorem. *Let Γ_{u} denote an affine deformation of Γ_0.*

- *The function*

$$\begin{aligned} \mathcal{C}_{\mathsf{per}}(\Sigma) &\longrightarrow \mathbb{R} \\ \mu_\gamma &\longmapsto \frac{\alpha(\gamma)}{\ell(\gamma)} \end{aligned}$$

  extends to a continuous function

$$\mathcal{C}(\Sigma) \xrightarrow{\Psi_{[\mathsf{u}]}} \mathbb{R}.$$

- *Γ_{u} acts properly if and only if $\Psi_{[\mathsf{u}]}(\mu) \neq 0$ for all $\mu \in \mathcal{C}(\Sigma)$.*

Since $\mathcal{C}(\Sigma)$ is connected, either $\Psi_{[\mathsf{u}]}(\mu) > 0$ for all $\mu \in \mathcal{C}(\Sigma)$ or $\Psi_{[\mathsf{u}]}(\mu) < 0$ for all $\mu \in \mathcal{C}(\Sigma)$. Compactness of $\mathcal{C}(\Sigma)$ implies that Margulis's invariant grows linearly with the length of γ:

Corollary. *For any $\mathsf{u} \in Z^1(\Gamma_0, \mathsf{V})$, there exists $C > 0$ such that*

$$|\alpha_{[\mathsf{u}]}(\gamma)| \leq C\ell(\gamma)$$

for all $\gamma \in \Gamma_{\mathsf{u}}$.

Margulis [28, 29] originally used the invariant α to detect properness: if Γ acts properly, then all the $\alpha(\gamma)$ are all positive, or all of them are negative. (Since conjugation by a homothety scales the invariant:

$$\alpha_{\lambda \mathsf{u}} = \lambda \alpha_{\mathsf{u}}$$

uniform positivity and uniform negativity are equivalent.) Goldman [20, 22] conjectured this necessary condition is sufficient: that is, properness is equivalent to the positivity (or negativity) of Margulis's invariant. Jones [23] proved this when Σ is a three-holed sphere. In this case the cone corresponding to proper actions is defined by the inequalities corresponding to the three boundary components $\gamma_1, \gamma_2, \gamma_3 \subset \partial\Sigma$:

$$\begin{aligned} \big\{[\mathsf{u}] \in & H^1(\Gamma_0, \mathsf{V}) \mid \alpha_{[\mathsf{u}]}(\gamma_i) > 0 \text{ for } i = 1,2,3\big\} \\ & \bigcup \big\{[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V}) \mid \alpha_{[\mathsf{u}]}(\gamma_i) < 0 \text{ for } i = 1,2,3\big\} \end{aligned}$$

However, when Σ is a one-holed torus, Charette [9] recently showed that the positivity of $\alpha_{[\mathsf{u}]}$ requires infinitely many inequalities, suggesting the original conjecture is generally false.

We thank Virginie Charette, David Fried, Todd Drumm, and Jonathan Rosenberg for helpful conversations. We are grateful to the hospitality of the Centre International de Recherches Mathématiques in Luminy where this paper was initiated. We are also grateful to Gena Naskov for several corrections.

Notation and Terminology

Denote the tangent bundle of a smooth manifold M by TM. For a given Riemannian metric on M, *the unit tangent bundle* UM consists of unit vectors in TM. For any subset $S \subset M$, denote the induced bundle over S by US. Denote the (real) hyperbolic plane by H^2.

Let $\mathbb{V}$ be a vector bundle over a manifold M. Denote by $\mathcal{R}^0(\mathbb{V})$ the vector space of sections of $\mathbb{V}$. Let ∇ be a connection on $\mathbb{V}$. If $\mathsf{s} \in \mathcal{R}^0(\mathbb{V})$ is a section and X is a vector field on M, then

$$\nabla_X(\mathsf{s}) \in \mathcal{R}^0(\mathbb{V}).$$

denotes covariant derivative of s with respect to X.

An affine bundle $\mathbb{E}$ over M is a bundle of affine spaces over M. The *vector bundle* $\mathbb{V}$ *associated to* $\mathbb{E}$ is the vector bundle whose fiber $\mathbb{V}_x$ over $x \in M$ is the vector bundle associated to the fiber $\mathbb{E}_x$. Denote the affine space of sections of E by $\mathcal{S}(\mathbb{E})$. If $s_1, s_2 \in \mathcal{S}(\mathbb{E})$ are two sections of E, the *difference* $s_1 - s_2$ is the section of $\mathbb{V}$, whose value at $x \in M$ is the translation $s_1(x) - s_2(x)$ of $\mathbb{E}_x$.

Denote the convex set of Borel probability measures on a topological space X by $\mathcal{P}(X)$. Denote the cohomology class of a cocycle z by $[z]$. A *flow* is an action of the additive group $\mathbb{R}$ of real numbers. The transformations in a flow ϕ are denoted ϕ_t, for $t \in \mathbb{R}$.

1. Affine geometry

This section collects general properties on affine spaces, affine transformations affine deformations of linear group actions. We are primarily interested in affine deformations of linear actions factoring through the irreducible $2r+1$-dimensional real representation V_r of

$$\mathsf{G}_0 \cong \mathsf{PSL}(2, \mathbb{R}),$$

where r is a positive integer.

1.1. **Affine spaces and their automorphisms.** Let V be a real vector space. Denote its group of linear automorphisms by $\mathsf{GL}(\mathsf{V})$.

An *affine space* E (modelled on V) is a space equipped with a simply transitive action of V. Call V the *vector space underlying* E, and refer to its elements as *translations.* Translation τ_{v} by a vector $\mathsf{v} \in \mathsf{V}$ is denoted by *addition:*

$$\begin{aligned} \mathsf{E} &\xrightarrow{\tau_{\mathsf{v}}} \mathsf{E} \\ x &\longmapsto x + \mathsf{v}. \end{aligned}$$

Let $x, y \in \mathsf{E}$. Denote the unique vector $\mathsf{v} \in \mathsf{V}$ such that $\tau_{\mathsf{v}}(x) = y$ by *subtraction:*

$$\mathsf{v} = y - x.$$

Let E be an affine space with associated vector space V. Choosing an arbitrary point $O \in \mathsf{E}$ (the *origin*) identifies E with V via the map

$$\begin{aligned} \mathsf{V} &\longrightarrow \mathsf{E} \\ \mathsf{v} &\longmapsto O + \mathsf{v}. \end{aligned}$$

An *affine automorphism* of E is the composition of a linear mapping (using the above identification of E with V) and a translation, that is,

$$\begin{aligned} \mathsf{E} &\xrightarrow{g} \mathsf{E} \\ O + \mathsf{v} &\longmapsto O + \mathbb{L}(g)(\mathsf{v}) + \mathsf{u}(g) \end{aligned}$$

which we simply write as

$$\mathsf{v} \longmapsto \mathbb{L}(g)(\mathsf{v}) + \mathsf{u}(g).$$

The affine automorphisms of E form a group $\mathsf{Aff}(\mathsf{E})$, and $(\mathbb{L}, \mathsf{u})$ defines an isomorphism of $\mathsf{Aff}(\mathsf{E})$ with the semidirect product $\mathsf{V} \rtimes \mathsf{GL}(\mathsf{V})$. The linear mapping $\mathbb{L}(g) \in \mathsf{GL}(\mathsf{V})$ is the *linear part* of the affine transformation g, and

$$\mathsf{Aff}(\mathsf{E}) \xrightarrow{\mathbb{L}} \mathsf{GL}(\mathsf{V})$$

is a homomorphism. The vector $\mathsf{u}(g) \in \mathsf{V}$ is the *translational part* of g. The mapping

$$\mathsf{Aff}(\mathsf{E}) \xrightarrow{\mathsf{u}} \mathsf{V}$$

satisfies a *cocycle identity:*

$$\mathsf{u}(\gamma_1\gamma_2) = \mathsf{u}(\gamma_1) + \mathbb{L}(\gamma_1)\mathsf{u}(\gamma_2) \tag{1.1}$$

for $\gamma_1, \gamma_2 \in \mathsf{Aff}(\mathsf{E})$.

1.2. **Affine deformations of linear actions.** Let $\Gamma_0 \subset \mathsf{GL}(\mathsf{V})$ be a group of linear automorphisms of a vector space V. Denote the corresponding Γ_0-module by V as well.

An *affine deformation* of Γ_0 is a representation

$$\Gamma_0 \xrightarrow{\rho} \mathsf{Aff}(\mathsf{E})$$

such that $\mathbb{L} \circ \rho$ is the inclusion $\Gamma_0 \hookrightarrow \mathsf{GL}(\mathsf{V})$. We confuse ρ with its image $\Gamma := \rho(\Gamma_0)$, which we also refer to as an *affine deformation* of Γ_0. Note that ρ embeds Γ_0 as the subgroup Γ of $\mathsf{GL}(\mathsf{V})$. In terms of the semidirect product decomposition

$$\mathsf{Aff}(\mathsf{E}) \cong \mathsf{V} \rtimes \mathsf{GL}(\mathsf{V})$$

an affine deformation is the graph $\rho = \rho_{\mathsf{u}}$ (with image denoted $\Gamma = \Gamma_{\mathsf{u}}$) of a *cocycle*

$$\Gamma_0 \xrightarrow{\mathsf{u}} V$$

that is, a map satisfying the cocycle identity (1.1). Write

$$\gamma = \rho(\gamma_0) = (\mathsf{u}(\gamma_0), \gamma_0) \in V \rtimes \Gamma_0$$

for the corresponding affine transformation:

$$x \xmapsto{\gamma} \gamma_0 x + \mathsf{u}(\gamma_0).$$

Cocycles form a vector space $Z^1(\Gamma_0, \mathsf{V})$. Cocycles $\mathsf{u}_1, \mathsf{u}_2 \in Z^1(\Gamma_0, \mathsf{V})$ are *cohomologous* if their difference $\mathsf{u}_1 - \mathsf{u}_2$ is a *coboundary,* a cocycle of the form

$$\Gamma_0 \xrightarrow{\delta \mathsf{v}_0} \mathsf{V}$$
$$\gamma \longmapsto \mathsf{v}_0 - \gamma \mathsf{v}_0.$$

Cohomology classes of cocycles form a vector space $H^1(\Gamma_0, \mathsf{V})$. Affine deformations $\rho_{\mathsf{u}_1}, \rho_{\mathsf{u}_2}$ are conjugate by translation by v_0 if and only if

$$\mathsf{u}_1 - \mathsf{u}_2 = \delta \mathsf{v}_0.$$

Thus $H^1(\Gamma_0, \mathsf{V})$ parametrizes translational conjugacy classes of affine deformations of $\Gamma_0 \subset \mathsf{GL}(\mathsf{V})$

An important affine deformation of Γ_0 is the *trivial affine deformation:* When $\mathsf{u} = 0$, the affine deformation Γ_{u} equals Γ_0 itself.

1.3. **Margulis's invariant of affine deformations.** Consider the case that $\mathsf{G}_0 = \mathsf{PSL}(2,\mathbb{R})$ and $\mathbb{L}$ is an irreducible representation of G_0. For every positive integer r, let $\mathbb{L}_r$ denote the irreducible representation of G_0 on the $2r$-symmetric power V_r of the standard representation of $\mathsf{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$. The dimension of V_r equals $2r+1$. The central element $-\mathbb{I} \in \mathsf{SL}(2,\mathbb{R})$ acts by $(-1)^{2r} = 1$, so this representation of $\mathsf{SL}(2,\mathbb{R}))$ defines a representation of $\mathsf{PSL}(2,\mathbb{R}) = \mathsf{SL}(2,\mathbb{R})/\{\pm\mathbb{I}\}$.

Furthermore the G_0-invariant nondegenerate skew-symmetric bilinear form on $\mathbb{R}^2$ induces a nondegenerate *symmetric* bilinear form B on V_r, which we normalize in the following paragraph.

An element $\gamma \in \mathsf{G}_0$ is *hyperbolic* if it corresponds to an element $\tilde{\gamma}$ of $\mathsf{SL}(2,\mathbb{R})$ with distinct real eigenvalues. Necessarily these eigenvalues are reciprocals λ, λ^{-1}, which we can uniquely specify by requiring $|\lambda| < 1$. Furthermore we choose eigenvectors $\mathsf{v}_+, \mathsf{v}_- \in \mathbb{R}^2$ such that:

- $\tilde{\gamma}(\mathsf{v}_+) = \lambda \mathsf{v}_+$;
- $\tilde{\gamma}(\mathsf{v}_-) = \lambda^{-1} \mathsf{v}_-$;
- The ordered basis $\{\mathsf{v}_-, \mathsf{v}_+\}$ is positively oriented.

Then the action $\mathbb{L}_r$ has eigenvalues λ^{2j}, for

$$j = -r, 1-r, \cdots -1, 0, 1, \dots, r-1, r,$$

where the symmetric product

$$\mathsf{v}_-^{r-j}\mathsf{v}_+^{r+j} \in \mathsf{V}_r$$

is an eigenvector with eigenvalue λ^{2j}. In particular γ fixes the vector

$$\mathsf{x}^0(\gamma) := c\mathsf{v}_-^r\mathsf{v}_+^r,$$

where the scalar c is chosen so that

$$\mathsf{B}(\mathsf{x}^0(\gamma), \mathsf{x}^0(\gamma)) = 1.$$

Call $\mathsf{x}^0(\gamma)$ the *neutral vector* of γ.

The subspaces

$$\mathsf{V}^-(\gamma) := \sum_{j=1}^{r} \mathbb{R}\big(\mathsf{v}_-^{r+j}\mathsf{v}_+^{r-j}\big)$$

$$\mathsf{V}^+(\gamma) := \sum_{j=1}^{r} \mathbb{R}\big(\mathsf{v}_-^{r-j}\mathsf{v}_+^{r+j}\big)$$

are γ-invariant and V enjoys a γ-invariant B-orthogonal direct sum decomposition

$$\mathsf{V} \;=\; \mathsf{V}^-(\gamma) \,\oplus\, \mathbb{R}\big(\mathsf{x}^0(\gamma)\big) \,\oplus\, \mathsf{V}^+(\gamma).$$

For any norm $\|\,\|$ on V, there exists $C, k > 0$ such that

$$\|\gamma^n(\mathsf{v})\| \leq Ce^{-kn}\|\mathsf{v}\| \text{for } \mathsf{v} \in \mathsf{V}^+(\gamma)$$
$$\|\gamma^{-n}(\mathsf{v})\| \leq Ce^{-kn}\|\mathsf{v}\| \text{for } \mathsf{v} \in \mathsf{V}^-(\gamma). \tag{1.2}$$

Furthermore

$$\mathsf{x}^0(\gamma^n) = |n|\mathsf{x}^0(\gamma)$$

if $n \in \mathbb{Z}, n \neq 0$, and

$$\mathsf{V}^{\pm}(\gamma^n) = \begin{cases} \mathsf{V}^{\pm}(\gamma) & \text{if } n > 0 \\ \mathsf{V}^{\mp}(\gamma) & \text{if } n < 0 \end{cases}.$$

For example, consider the hyperbolic one-parameter subgroup A_0 comprising $a(t)$ where

$$a(t) : \begin{cases} \mathsf{v}_+ & \longmapsto e^{t/2}\mathsf{v}_+ \\ \mathsf{v}_- & \longmapsto e^{-t/2}\mathsf{v}_- \end{cases}$$

In that case the action on V_1 corresponds to the one-parameter group of isometries of $\mathbb{R}^{2,1}$ defined by

$$a(t) \longmapsto \begin{bmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & 1 & 0 \\ \sinh(t) & 0 & \cosh(t) \end{bmatrix}$$

with neutral vector

$$\mathsf{v}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Next suppose that $g \in \mathsf{Aff}(\mathsf{E})$ is an affine transformation whose linear part $\gamma = \mathbb{L}(\gamma)$ is hyperbolic. Then there exists a unique affine line $l = l_g \subset E$ which is g-invariant. The line l is parallel to the neutral vector $\mathsf{x}^0(\gamma)$. The restriction of g to l_g is a translation by

$$\mathsf{B}\big(\, gx - x \,,\, \mathsf{x}^0(\gamma)\,\big)\ \ \mathsf{x}^0(\gamma)$$

where $\mathsf{x}^0(\gamma)$ is chosen so that $\mathsf{B}(\mathsf{x}^0(\gamma), \mathsf{x}^0(\gamma)) = 1$.

Suppose that $\Gamma_0 \subset \mathsf{G}_0$ be a *Schottky group,* that is, a nonabelian discrete subgroup containing only hyperbolic elements. Such a discrete subgroup is a free group of rank at least two. The adjoint representation of G_0 defines an isomorphism of G_0 with the identity component $\mathsf{O}(2,1)^0$ of the orthogonal group of the 3-dimensional Lorentzian vector space $\mathbb{R}^{2,1}$.

Let $\mathsf{u} \in Z^1(\Gamma_0, \mathsf{V})$ be a cocycle defining an affine deformation ρ_{u} of Γ_0. In [28, 29], Margulis constructed the invariant

$$\begin{aligned} \Gamma_0 &\xrightarrow{\alpha_{\mathsf{u}}} \mathbb{R} \\ \gamma &\longmapsto \mathsf{B}(\mathsf{u}(\gamma), \mathsf{x}^0(\gamma)). \end{aligned}$$

This well-defined class function on Γ_0 satisfies

$$\alpha_{\mathsf{u}}(\gamma^n) = |n| \alpha_{\mathsf{u}}(\gamma)$$

(when r is odd) and depends only the cohomology class $[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V})$. Furthermore $\alpha_{\mathsf{u}}(\gamma) = 0$ if and only if $\rho_{\mathsf{u}}(\gamma)$ fixes a point in E. Two affine deformations of a given Γ_0 are conjugate if and only if they have the same Margulis invariant (Drumm-Goldman [18]). An affine deformation γ_{u} of Γ_0 is *radiant* if it satisfies any of the following equivalent conditions:

- Γ_{u} fixes a point;
- Γ_{u} is conjugate to Γ_0;
- The cohomology class $[\mathsf{u}] \in H^1(\Gamma_0, \mathsf{V})$ is zero;
- The Margulis invariant α_{u} is identically zero.

For further discussion see [9, 10, 17, 20, 22, 27].)

The centralizer of Γ_0 in the general linear group $\mathsf{GL}(\mathbb{R}^3)$ consists of *homotheties*

$$\mathsf{v} \xmapsto{h_\lambda} \lambda \mathsf{v}$$

where $\lambda \neq 0$. The homothety h_λ conjugates an affine deformation (ρ_0, u) to $(\rho_0, \lambda\mathsf{u})$. Thus conjugacy classes of non-radiant affine deformations are parametrized by the projective space $\mathbb{P}(H^1(\Gamma_0, \mathsf{V}))$. Our main result is that conjugacy classes of proper actions comprise a convex domain in $\mathbb{P}(H^1(\Gamma_0, \mathsf{V}))$.

2. Flat bundles associated to affine deformations

We define two fiber bundles over $U\Sigma$. The first bundle $\mathbb{V}$ is a *vector bundle* associated to the original group Γ_0. The second bundle $\mathbb{E}$ is an *affine bundle* associated to the affine deformation Γ. The vector bundle underlying $\mathbb{E}$ is $\mathbb{V}$. The vector bundle $\mathbb{V}$ has a flat linear connection and the affine bundle $\mathbb{E}$ has a flat affine connection, each denoted ∇. (For the theory of connections on affine bundles, see Kobayashi-Nomizu [25].) The vector field Ξ tangent to the flow Φ defined above identifies with the unique vector field on the total space of $\mathbb{E}$ which is ∇-horizontal and covers the vector field ξ defining the geodesic flow on $U\Sigma$.

2.1. **Semidirect products and homogeneous affine bundles.** Consider a Lie group G_0, a vector space V, and a linear representation $\mathsf{G}_0 \xrightarrow{\mathbb{L}} \mathsf{GL}(\mathsf{V})$. Let G be the corresponding semidirect product $\mathsf{V} \rtimes \mathsf{G}_0$. Multiplication in G is defined by

$$(2.1) \qquad (\mathsf{v}_1, g_1) \cdot (\mathsf{v}_2, g_2) \ := \ (\mathsf{v}_1 + \mathbb{L}(g_1)\mathsf{v}_2, g_1 g_2).$$

Projection $\mathsf{G} \xrightarrow{\Pi} \mathsf{G}_0$ defines a trivial bundle with fiber V over G_0. It is equivariant with respect to the action of G on the total space by left-multiplication and the action of G on the base obtained from left-multiplication on G_0 and the homomorphism $\mathbb{L}$. Since $\mathbb{L}$ is a homomorphism, (2.1) implies equivariance of Π.

Consider Π as a (trivial) *affine bundle* by disregarding the origin in V. This affine structure is G-invariant: For (2.1) implies that the action of $\gamma_1 = (\mathsf{v}_1, g_1)$ on the total space covers the action of g_1 on the base. On the fibers the action is affine with linear part $g_1 = \mathbb{L}(\gamma_1)$ and translational part $\mathsf{v}_1 = \mathsf{u}(\gamma_1)$. Denote this G-homogeneous affine bundle over G_0 by $\tilde{\mathbb{E}}$.

Consider Π also as a (trivial) *vector bundle.* By (2.1), this structure is G_0-invariant. Via $\mathbb{L}$, this G_0-homogeneous vector bundle becomes a G-homogeneous vector bundle $\tilde{\mathbb{V}}$.

This G*-homogeneous vector bundle underlies* $\tilde{\mathbb{E}}$*:* Let $(\mathsf{v}_2' - \mathsf{v}_2, 1)$ be the translation taking (v_2, g_2) to (v_2', g_2). Then (2.1) implies that $\gamma_1 = (\mathsf{v}_1, g_1)$ acts on $(\mathsf{v}_2' - \mathsf{v}_2, g_2)$ by $\mathbb{L}$:

$$\big((\mathsf{v}_1 + g_1(\mathsf{v}_2')) - (\mathsf{v}_1 + g_1(\mathsf{v}_2)), g_1 g_2\big) \ = \ \big(\mathbb{L}(g_1)(\mathsf{v}_2' - \mathsf{v}_2), g_1 g_2\big).$$

In our examples, $\mathbb{L}$ preserves a a bilinear form B on V. The G_0-invariant bilinear form $V \times V \xrightarrow{\mathsf{B}} \mathbb{R}$ defines a bilinear pairing $\tilde{\mathbb{V}} \times \tilde{\mathbb{V}} \xrightarrow{\mathbb{B}} \mathbb{R}$ of vector bundles.

2.2. **Homogeneous connections.** The G-homogeneous affine bundle $\tilde{\mathbb{E}}$ and the G-homogeneous vector bundle $\tilde{\mathbb{V}}$ admit flat connections (each denoted $\tilde{\nabla}$) as follows. To specify $\tilde{\nabla}$, it suffices to define the covariant derivative of a section $\tilde{s}$ over a smooth path $g(t)$ in the base. For either $\tilde{\mathbb{E}} = G$ or $\tilde{\mathbb{V}} = G$, a section is determined by a smooth path $\mathsf{v}(t) \in \mathsf{V}$:

$$\begin{aligned} \mathbb{R} &\xrightarrow{\tilde{s}} \mathsf{V} \rtimes \mathsf{G}_0 = \mathsf{G} \\ t &\longmapsto (\mathsf{v}(t), g(t)). \end{aligned}$$

Define

$$\frac{D}{dt}\tilde{s}(t) := \frac{d}{dt}\mathsf{v}(t) \in \mathsf{V}.$$

If now $\tilde{s}$ is a section of $\tilde{\mathbb{E}}$ or $\tilde{\mathbb{V}}$, and X is a tangent vector field, define

$$\tilde{\nabla}_X(\tilde{s}) = \frac{D}{dt}\tilde{s}(g(t))$$

where $g(t)$ is any smooth path with

$$g'(t) = X\big(g(t)\big).$$

The resulting covariant differentiation operators define connections on $\mathbb{V}$ and $\mathbb{E}$ which are invariant under the respective G-actions.

2.3. **Flatness.** For each $\mathsf{v} \in \mathsf{V}$,

$$\begin{aligned} \mathsf{G}_0 &\xrightarrow{\tilde{s}_{\mathsf{v}}} \mathsf{V} \rtimes \mathsf{G}_0 = \mathsf{G} \\ g &\longmapsto (\mathsf{v}, g) \end{aligned}$$

defines a section whose image is the coset $\{\mathsf{v}\} \times \mathsf{G}_0 \subset \mathsf{G}$. Clearly these sections are parallel with respect to $\tilde{\nabla}$. Since the sections $\tilde{s}_{\mathsf{v}}$ foliate G, the connections are flat.

If $\mathbb{L}$ preserves a bilinear form B on V, the bilinear pairing $\mathbb{B}$ on $\tilde{\mathbb{V}}$ is parallel with respect to $\tilde{\nabla}$.

3. Sections and subbundles

Now we describe the sections and subbundles of the homogeneous bundles over $\mathsf{G}_0 \cong \mathsf{PSL}(2,\mathbb{R})$ associated to the irreducible $2r+1$-dimensional representation V_r.

3.1. **The flow on the affine bundle.** Right-multiplication by $a(t)$ on G defines a flow $\tilde{\Phi}_t$ on $\tilde{\mathbb{E}}$. Since $\mathsf{A}_0 \subset \mathsf{G}_0$, this flow covers the flow ϕ_t on G_0 defined by right-multiplication by $a(t)$ on G_0. That is, the diagram

$$\begin{array}{ccc} \tilde{\mathbb{E}} & \xrightarrow{\tilde{\Phi}_t} & \tilde{\mathbb{E}} \\ {\scriptstyle\tilde{\Pi}}\big\downarrow & & \big\downarrow{\scriptstyle\tilde{\Pi}} \\ U\mathsf{H}^2 & \xrightarrow[\tilde{\phi}_t]{} & U\mathsf{H}^2 \end{array}$$

commutes. The vector field $\tilde{\Phi}$ on $\tilde{\mathbb{E}}$ generating this flow covers the vector field $\tilde{\phi}$ generating ϕ_t.

Furthermore $\tilde{s}_v(ga(-t)) = \tilde{s}(v)a(-t)$ implies $\tilde{s}_v \circ \tilde{\phi}_t = \tilde{\Phi}_t \circ \tilde{s}_v$, whence $\tilde{\Xi}$ is the $\tilde{\nabla}$-*horizontal lift* of $\tilde{\xi}$.

The flow $\tilde{\Phi}_t$ commutes with the action of G. Thus $\tilde{\Phi}_t$ *is a flow on the flat* G*-homogeneous affine bundle* $\tilde{\mathbb{E}}$ *covering* ϕ_t.

Right-multiplication by $a(t)$ on G also defines a flow on the flat G-homogeneous vector bundle $\tilde{\mathbb{V}}$ covering $\tilde{\phi}_t$. Identifying $\tilde{\mathbb{V}}$ as the vector bundle underlying $\tilde{\mathbb{E}}$, the $\mathbb{R}$-action is just the linearization $D\tilde{\Phi}_t$ of the action $\tilde{\Phi}_t$:

$$\begin{array}{ccc} \tilde{\mathbb{V}} & \xrightarrow{D\tilde{\Phi}_t} & \tilde{\mathbb{V}} \\ \downarrow & & \downarrow \\ U\mathsf{H}^2 & \xrightarrow[\tilde{\phi}_t]{} & U\mathsf{H}^2 \end{array}$$

3.2. **The neutral section.** The G-action and the flow $D\tilde{\Phi}_t$ on $\tilde{\mathbb{V}}$ preserve a section $\tilde{\nu} \in \tilde{\mathbb{V}}$ defined as follows. The one-parameter subgroup A_0 fixes the neutral vector $\mathsf{v}_0 \in \mathsf{V}$. Let $\tilde{\nu}$ denote the section of $\tilde{\mathbb{V}}$ defined by:

$$\begin{aligned} U\mathsf{H}^2 \approx \mathsf{G}_0 &\xrightarrow{\tilde{\nu}} \mathsf{V} \rtimes \mathsf{G}_0 \approx \tilde{\mathbb{V}} \\ g &\longmapsto (\mathbb{L}(g)\mathsf{v}_0, g). \end{aligned}$$

In terms of the group operation on G, this section is given by *right-multiplication* by $\mathsf{v}_0 \in V \subset \mathsf{G}$ acting on $g \in \mathsf{G}_0 \subset \mathsf{G}$. Since $\mathbb{L}$ is a homomorphism,

$$\tilde{\nu}(hg) = h\tilde{\nu}(g),$$

so $\tilde{\nu}$ defines a G-invariant section of $\tilde{\mathbb{V}}$.

Although $\tilde{\nu}$ is not parallel in every direction, it is parallel along the flow $\tilde{\Phi}_t$:

Lemma 3.1. $\tilde{\nabla}_{\tilde{\xi}}(\tilde{\nu}) = 0$.

Proof. Let $g \in \mathsf{G}_0$. Then

$$\tilde{\nabla}_{\tilde{\xi}}(\tilde{\nu})(g) = \frac{D}{dt}\bigg|_{t=0} \tilde{\nu}(\tilde{\phi}_t(g)) = \frac{D}{dt}\bigg|_{t=0} ga(-t)\mathsf{v}_0 = 0$$

since $a(-t)\mathsf{v}_0 = \mathsf{v}_0$ is constant. ☐

The section $\tilde{\nu}$ is the diffuse analogue of the neutral eigenvector $\mathsf{x}^0(\gamma)$ of a hyperbolic element $\gamma \in \mathsf{O}(2,1)^0$ discussed in §1.3. Another approach to the flat connection ∇ and the neutral section ν is given in Labourie [27].

3.3. **Stable and unstable subbundles.** Let $\mathsf{V}^0 \subset \mathsf{V}$ denote the line of vectors fixed by A_0, that is the line spanned by v_0. The eigenvalues of $a(t)$ acting on V are the $2r+1$ distinct positive real numbers

$$e^r, e^{r-2}, \dots, 1, \dots, e^{2-r}, e^{-r}$$

and the eigenspace decomposition of V is invariant under $a(t)$. Let V^+ denote the sum of eigenspaces for eigenvalues > 1 and V^- denote the sum of eigenspaces for eigenvalues < 1. The corresponding decomposition

$$\mathsf{V} = \mathsf{V}^- \oplus \mathsf{V}^0 \oplus \mathsf{V}^+$$

is $a(t)$-invariant and defines a (left)G-invariant decomposition of the vector bundle $\tilde{\mathbb{V}}$ into subspaces which are invariant under $D\tilde{\Phi}_t$.

4. Hyperbolic geometry

Let $\mathbb{R}^{2,1}$ be the 3-dimensional real vector space with inner product

$$B(\mathsf{u},\mathsf{v}) := \mathsf{u}_1\mathsf{v}_1 + \mathsf{u}_2\mathsf{v}_2 - \mathsf{u}_3\mathsf{v}_3$$

and $\mathsf{G}_0 = \mathsf{O}(2,1)^0$ the identity component of its group of isometries. G_0 consists of linear isometries of $\mathbb{R}^{2,1}$ which preserve both an orientation of $\mathbb{R}^{2,1}$ and a *time-orientation* (or *future*) of $\mathbb{R}^{2,1}$, that is, a connected component of the *open light-cone*

$$\{\mathsf{v} \in \mathbb{R}^{2,1} \mid B(\mathsf{v},\mathsf{v}) < 0\}.$$

Then

$$\mathsf{G}_0 = \mathsf{O}(2,1)^0 \cong \mathsf{PSL}(2,\mathbb{R}) \cong \mathsf{Isom}^0(\mathsf{H}^2)$$

where H^2 denotes the real hyperbolic plane. The representation $\mathbb{R}^{2,1}$ is isomorphic to the irreducible representation V_2 defined in §1.3.

4.1. **The hyperboloid model.** We define H^2 as follows. We work in the irreducible representation $\mathbb{R}^{2,1} = \mathsf{V}_1$ (isomorphic to the adjoint representation of G_0). Let B denote the invariant bilinear form (isomorphic to the Killing form). The two-sheeted hyperboloid

$$\{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathsf{B}(\mathsf{v},\mathsf{v}) = -1\}$$

has two connected components. If v_1 lies in this hyperboloid, then its connected component equals

$$\{\mathsf{v} \in \mathbb{R}^{2,1} \mid \mathsf{B}(\mathsf{v},\mathsf{v}) = -1, \mathsf{B}(\mathsf{v},\mathsf{v}_1) < 0\}. \tag{4.1}$$

For clarity, we fix a timelike vector v_1, for example

$$\mathsf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{2,1},$$

and define H^2 as the connected component (4.1).

The Lorentzian metric defined by B restricts to a Riemannian metric of constant curvature -1 on H^2. The identity component G_0 of $\mathsf{O}(2,1)^0$ is the group $\mathsf{Isom}^0(\mathbb{E}^{2,1})$ of orientation-preserving isometries of H^2. The stabilizer of v_1

$$\mathsf{K}_0 := \mathsf{PO}(2).$$

is the *maximal compact subgroup* of G_0. Evaluation at v_1

$$\begin{aligned} \mathsf{G}_0/\mathsf{K}_0 &\longrightarrow \mathbb{R}^{2,1} \\ g\mathsf{K}_0 &\longmapsto g\mathsf{v}_1 \end{aligned}$$

identifies H^2 with the homogeneous space $\mathsf{G}_0/\mathsf{K}_0$.

The action of G_0 canonically lifts to a *simply transitive* (left-) action on the unit tangent bundle $U\mathsf{H}^2$. The unit spacelike vector

$$\dot{\mathsf{v}}_1 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2,1} \tag{4.2}$$

defines a unit tangent vector to H^2 at v_1, which we denote:

$$(\mathsf{v}_1, \dot{\mathsf{v}}_1) \in U\mathsf{H}^2.$$

Evaluation at $(\mathsf{v}_1, \dot{\mathsf{v}}_1)$

$$\begin{aligned} \mathsf{G}_0 &\xrightarrow{\mathcal{E}} U\mathsf{H}^2 \\ g_0 &\longmapsto g_0(\mathsf{v}_1, \dot{\mathsf{v}}_1) \end{aligned}$$

G_0-equivariantly identifies G_0 with the unit tangent bundle $U\mathsf{H}^2$. Here G_0 acts on itself by left-multiplication. The action on $U\mathsf{H}^2$ is the action induced by isometries of H^2. Under this identification, K_0 corresponds to the fiber of $U\mathsf{H}^2$ above v_1.

4.2. **The geodesic flow.** The *Cartan subgroup*

$$\mathsf{A}_0 := \mathsf{PO}(1,1).$$

is the one-parameter subgroup comprising

$$a(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{bmatrix}$$

for $t \in \mathbb{R}$. Under this identification,

$$a(t)\mathsf{v}_1 \in \mathsf{H}^2$$

describes the geodesic through v_1 tangent to $\dot{\mathsf{v}}_1$.

Right-multiplication by $a(t)$ on G_0 identifies with the geodesic flow $\tilde{\phi}_t$ on $U\mathsf{H}^2$. First note that $\tilde{\phi}_t$ on $U\mathsf{H}^2$ commutes with the action of G_0 on $U\mathsf{H}^2$, which corresponds to the action on G_0 on itself by left-multiplications. The $\mathbb{R}$-action on G_0 corresponding to $\tilde{\phi}_t$ commutes with left-multiplications on G_0. Thus this $\mathbb{R}$-action on G_0 must be right-multiplication by a one-parameter subgroup. At the basepoint $(\mathsf{v}_1, \dot{\mathsf{v}}_1)$, the geodesic flow corresponds to by right-multiplication by

A_0. Hence this one-parameter subgroup must be A_0. Therefore right-multiplication by A_0 on G_0 induces the geodesic flow $\tilde{\phi}_t$ on $U\mathsf{H}^2$.

Denote the vector field corresponding to the geodesic flow by $\tilde{\xi}$:

$$\tilde{\xi}(x) := \frac{d}{dt}\bigg|_{t=0} \tilde{\phi}_t(x).$$

4.3. **The convex core.** Since Γ_0 is a Schottky group, the complete hyperbolic surface is *convex cocompact.* That is, there exists a compact geodesically convex subsurface $\mathsf{core}(\Sigma)$ such that the inclusion

$$\mathsf{core}(\Sigma) \subset \Sigma$$

is a homotopy equivalence. $\mathsf{core}(\Sigma)$ is called the *convex core* of Σ, and is bounded by closed geodesics $\partial_i(\Sigma)$ for $i = 1, \dots, k$. Equivalently the discrete subgroup Γ_0 is finitely geneerated and contains only hyperbolic elements.

Another criterion for convex cocompactness involves the *ends* $e_i(\Sigma)$ of Σ. Each component $e_i(\Sigma)$ of the closure of $\Sigma \setminus \mathsf{core}(\Sigma)$ is diffeomorphic to a product

$$e_i(\Sigma) \xrightarrow{\approx} \partial_i(\Sigma) \times [0, \infty).$$

The corresponding end of $U\Sigma$ is diffeomorphic to the product

$$e_i(U\Sigma) \xrightarrow{\approx} \partial_i(\Sigma) \times S^1 \times [0, \infty).$$

The corresponding Cartesian projections

$$e_i(\Sigma) \longrightarrow \partial_i(\Sigma)$$

and the identity map on $\mathsf{core}(\Sigma)$ extend to a deformation retraction

$$\Sigma \xrightarrow{\Pi_{\mathsf{core}(\Sigma)}} \mathsf{core}(\Sigma).$$

We use the following standard compactness results (see Kapovich [24], §4.17 (pp.98–102) or Canary-Epstein-Green [8] for discussion and proof):

Lemma 4.1. *Let $\Sigma = \mathsf{H}^2/\Gamma_0$ be the hyperbolic surface where Γ_0 is convex cocompact and let $R \geq 0$. Then*

$$\mathcal{K}_R := \{ y \in \Sigma \mid d(y, \mathsf{core}(\Sigma)) \leq R \}$$

is a compact geodesically convex subsurface, upon which the restriction of $\Pi_{\mathsf{core}(\Sigma)}$ is a homotopy equivalence.

Lemma 4.2. *Let Σ be as above, $U\Sigma$ its unit tangent bundle and ϕ_t the geodesic flow on $U\Sigma$. Then the space of ϕ-invariant Borel probability measures on $U\Sigma$ is a convex compact space $\mathcal{C}(\Sigma)$ with respect to the weak $\star$-topology.*

Proof. The subbundle $U\mathsf{core}(\Sigma)$ comprising unit vectors over points in $\mathsf{core}(\Sigma)$ is compact.

Any geodesic which exits $\mathsf{core}(\Sigma)$ into an end $e_i(\Sigma)$ remains in the same end $e_i(\Sigma)$. Therefore every recurrent geodesic ray eventually lies in $\mathsf{core}(\Sigma)$. By the Poincaré recurrence theorem, every ϕ-invariant probability measure on $U\Sigma$ is supported over $\mathsf{core}(\Sigma)$. Indeed, the union of all recurrent orbits of ϕ is a closed ϕ-invariant subset $U_{\mathsf{rec}}\Sigma \subset U\Sigma$ which we now show is compact.

On the unit tangent bundle $U\mathsf{H}^2$, every ϕ orbit tends to a unique ideal point on $S^1_\infty = \partial\mathsf{H}^2$; let

$$U\mathsf{H}^2 \xrightarrow{\eta} S^1_\infty$$

denote the corresponding map. Then the ϕ-orbit of $\tilde{x} \in U\mathsf{H}^2$ defines a geodesic which is recurrent in the forward direction if and only if $\eta(\tilde{x}) \in \Lambda$, where $\Lambda \subset S^1_\infty$ is the *limit set* of Γ_0. The geodesic is recurrent in the backward direction if $\eta(-\tilde{x}) \in \Lambda$. Then every recurrent orbit of the geodesic flow lies in the compact set $U_{\mathsf{rec}}\Sigma$ consisting of vectors $x \in U\mathsf{core}(\Sigma)$ with

$$\eta(\tilde{x}), \eta(-\tilde{x}) \in \Lambda.$$

Since $U_{\mathsf{rec}}\Sigma$ is compact, $\mathcal{P}(U_{\mathsf{rec}}\Sigma)$ is a compact convex set, and the closed subset of ϕ-invariant probability measures on $U_{\mathsf{rec}}\Sigma$ is also a compact convex set. Since every ϕ-invariant probability measure on $U\Sigma$ is supported on $U_{\mathsf{rec}}\Sigma$, the assertion follows. □

4.4. **Bundles over $U\Sigma$.** Let $\Gamma \subset \mathsf{G}$ be an affine deformation of a discrete subgroup $\Gamma_0 \subset \mathsf{G}_0$. Since Γ is a discrete subgroup of G, the quotient $\mathbb{E} := \tilde{\mathbb{E}}/\Gamma$ is an affine bundle over $U\Sigma = U\mathsf{H}^2/\Gamma_0$ and inherits a flat connection ∇ from the flat connection $\tilde{\nabla}$ on $\tilde{\mathbb{E}}$. Furthermore the flow $\tilde{\Phi}_t$ on $\tilde{\mathbb{E}}$ descends to a flow Φ on $\mathbb{E}$ which is the horizontal lift of the flow ϕ on $U\Sigma$.

The vector bundle $\mathbb{V}$ underlying $\mathbb{E}$ is the quotient

$$\mathbb{V} := \tilde{\mathbb{V}}/\Gamma = \tilde{\mathbb{V}}/\Gamma_0$$

and inherits a flat linear connection ∇ from the flat linear connection $\tilde{\nabla}$ on $\tilde{\mathbb{V}}$. The flow $D\tilde{\Phi}_t$ on $\tilde{\mathbb{V}}$ covering $\tilde{\phi}_t$, the neutral section $\tilde{\nu}$, and the stable-unstable splitting all descend to a flow $D\Phi$, a section ν and a splitting

$$\mathbb{V} = \mathbb{V}^- \oplus \mathbb{V}^0 \oplus \mathbb{V}^+ \tag{4.3}$$

of the flat vector bundle $\mathbb{V}$ over $U\Sigma$. In particular

$$\big(\Pi_{\mathsf{core}(\Sigma)}\big)^* \mathbb{V}_{\mathsf{core}(\Sigma)} \cong \mathbb{V}$$

since $\Pi_{\mathsf{core}(\Sigma)}$ is a homotopy equivalence.

We choose a Euclidean metric g on $\mathbb{V}$ with the following properties:

- The neutral section ν is bounded with respect to g;
- The flat linear connection ∇ is bounded with respect to g;
- The flow $D\phi$ exponentially expands the subbundle V^+ and exponentially contracts V^-.

One may construct such a metric by choosing any Euclidean inner product on the vector space V and extending it to a Euclidean metric on the vector bundle

$$\tilde{\mathbb{V}} \approx G = G_0 \times \mathsf{V}$$

which is invariant under left-multiplications in G_0. This Euclidean metric satisfies the following *hyperbolicity* condition: Constants $C, k > 0$ exist so that

$$\|D(\phi_t)\mathsf{v}\| \leq Ce^{kt}\|\mathsf{v}\| \quad \text{as } t \longrightarrow -\infty,$$

for $\mathsf{v} \in \mathbb{V}^+$, and

$$\|D(\phi_t)\mathsf{v}\| \leq Ce^{-kt}\|\mathsf{v}\| \quad \text{as } t \longrightarrow +\infty.$$

for $\mathsf{v} \in \mathbb{V}^-$, which follow immediately from (1.2).

4.5. **Proper Γ-actions and proper $\mathbb{R}$-actions.** Let X be a locally compact Hausdorff space with homeomorphism group $\mathsf{Homeo}(X)$. Suppose that H is a closed subgroup of of $\mathsf{Homeo}(X)$ with respect to the compact-open topology. Recall that H acts *properly* if the mapping

$$\begin{aligned} H \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

is a proper mapping. That is, for every pair of compact subsets $K_1, K_2 \subset X$, the set

$$\{g \in H \mid gK_1 \cap K_2 \neq \emptyset\}$$

is compact. The usual notion of a *properly discontinuous* action is a proper action where H is given the discrete topology. In this case, the quotient X/H is a Hausdorff space. If H acts freely, then the quotient mapping $X \longrightarrow X/H$ is a covering space. For background on proper actions, see Bourbaki [7], Koszul [26] and Palais [32].

The question of properness of the Γ-action is equivalent to that of properness of an action of $\mathbb{R}$.

Proposition 4.3. *An affine deformation Γ acts properly on E if and only if A_0 acts properly by right-multiplication on G/Γ.*

The following lemma is a key tool in our argument. (A related statement is proved in Benoist [5], Lemma 3.1.1. For a proof in a different context see Rieffel [33, 34].)

Lemma 4.4. *Let X be a locally compact Hausdorff space. Let A and B be commuting groups of homeomorphisms of X, each of which acts properly on X. Then the following conditions are equivalent:*

(1) *A acts properly on X/B;*
(2) *B acts properly on X/A;*
(3) *$A \times B$ acts properly on X.*

Proof. We prove (3) $\Longrightarrow$ (1). Suppose $A \times B$ acts properly on X but A does not act properly on X/B. Then a B-invariant subset $H \subset X$ exists such that:

- $H/B \subset X/B$ is compact;
- The subset
$$A_{H/B} := \{a \in A \mid a(H/B) \cap (H/B) \neq \emptyset\}$$
is not compact.

We claim a compact subset $K \subset X$ exists satisfying $B \cdot K = H$.

Denote the quotient mapping by
$$X \xrightarrow{\Pi_B} X/B.$$
For each $x \in \Pi_B^{-1}(H/B)$, choose a precompact open neighborhood $U(x) \subset X$. The images $\Pi_B(U(x))$ define an open covering of H/B. Choose a finite subset $x_1, \dots, x_l$ such that the open sets $\Pi_B(U(x_i))$ cover H/B. Then the union
$$K' := \bigcup_{i=1}^{l} \overline{U(x_i)}$$
is a compact subset of X such that $\Pi_B(K') \supset H/B$. Taking $K = K' \cap \Pi_B^{-1}(H/B)$ proves the claim.

Since A acts properly on X, the subset
$$(A \times B)_K := \{(a, b) \in A \times B \mid aK \cap bK \neq \emptyset\}$$
is compact. But $A_{H/B}$ is the image of the compact set $(A \times B)_K$ under Cartesian projection $A \times B \longrightarrow A$ and is compact, a contradiction.

We prove that (1) $\Longrightarrow$ (3). Suppose that A acts properly on X/B and $K \subset X$ is compact. Then Cartesian projection $A \times B \longrightarrow A$ maps
$$(A \times B)_K \longrightarrow A_{(B \cdot K)/B}. \tag{4.4}$$
A acts properly on X/B implies $A_{(B \cdot K)/B}$ is a compact subset of A. But (4.4) is a proper map, so $(A \times B)_K$ is compact, as desired.

Thus (3) $\Longleftrightarrow$ (1). The proof (3) $\Longleftrightarrow$ (2) is similar. $\square$

Proof of Proposition 4.3. Apply Lemma 4.4 with $X = \mathsf{G}$ and A to be the action of $A_0 \cong \mathbb{R}$ by right-multiplication and B to be the action of Γ by left-multiplication. The lemma implies that Γ acts properly on $\mathsf{E} = \mathsf{G}/\mathsf{G}_0$ if and only if G_0 acts properly on $\Gamma\backslash\mathsf{G}$.

Apply the Cartan decomposition $\mathsf{G}_0 = \mathsf{K}_0\mathsf{A}_0\mathsf{K}_0$. Since K_0 is compact, the action of G_0 on E is proper if and only if its restriction to A_0 is proper. ☐

5. Labourie's diffusion of Margulis's invariant

In [27], Labourie defined a function $U\Sigma \xrightarrow{F_{\mathsf{u}}} \mathbb{R}$, corresponding to the invariant α_{u} defined by Margulis [28, 29]. Margulis's invariant $\alpha = \alpha_{\mathsf{u}}$ is an $\mathbb{R}$-valued class function on Γ_0 whose value on $\gamma \in \Gamma_0$ equals

$$B\big(\, \rho_{\mathsf{u}}(\gamma)O - O \,,\, \mathsf{x}^0(\gamma)\,\big)$$

where $\mathsf{x}^0(\gamma) \in V$ is the neutral vector of γ (see §1.3 and the references listed there).

Now the origin $O \in E$ will be replaced by a section s of $\mathbb{E}$, the vector $\mathsf{x}^0(\gamma) \in V$ will be replaced by the neutral section ν of $\mathbb{V}$, and the linear action of Γ_0 will be replaced by the geodesic flow ϕ_t on $U\Sigma$.

Let s be a section of $\mathbb{E}$. Its covariant derivative with respect to ξ is a section $\mathbb{V}.\nabla_\xi(s) \in \mathcal{R}^0(\mathbb{V})$. Pairing with $\nu \in \mathcal{R}^0(\mathbb{V})$ via

$$\mathbb{V} \times \mathbb{V} \xrightarrow{\mathbb{B}} \mathbb{R}$$

produces a function $U\Sigma \xrightarrow{F_{\mathsf{u},s}} \mathbb{R}$:

$$F_{\mathsf{u},s} := \mathbb{B}(\nabla_\xi(s), \nu).$$

(Compare Labourie [27].)

5.1. **The invariant is well-defined.**

Henceforth choose a Riemannian metric g on $\mathbb{V}$ such that B and ν are bounded with respect to g. Let $\mathcal{S}(\mathbb{E})$ denote the space of continuous sections s of $\mathbb{E}$, differentiable along ξ, such that $\nabla_\xi(s)$ is bounded with respect to g. In particular $F_{\mathsf{u},s}$ is bounded and measurable.

Lemma 5.1. *Let $s \in \mathcal{S}(\mathbb{E})$ and μ be a ϕ-invariant Borel probability measure on $U\Sigma$. Then*

$$\int_{U\Sigma} F_{\mathsf{u},s}\, d\mu$$

is independent of s.

Proof. Let $s_1, s_2 \in \mathcal{S}(\mathbb{E})$ be sections. Then $\mathsf{s} = s_1 - s_2$ is a section of $\mathbb{V}$ and

$$\nabla_\xi(s_1) - \nabla_\xi(s_2) = \nabla_\xi(\mathsf{s}).$$

Therefore

$$\int_{U\Sigma} F_{\mathsf{u},s_1} d\mu - \int_{U\Sigma} F_{\mathsf{u},s_2} d\mu = \int_{U\Sigma} \mathbb{B}(\nabla_\xi(\mathsf{s}),\nu)d\mu$$
$$= \int_{U\Sigma} \xi\mathbb{B}(\mathsf{s},\nu)d\mu - \int_{U\Sigma} \mathbb{B}(\mathsf{s},\nabla_\xi(\nu))d\mu.$$

The first term vanishes since μ is ϕ-invariant:

$$\int_{U\Sigma} \xi\mathbb{B}(\mathsf{s},\nu)d\mu = \int_{U\Sigma} \frac{d}{dt}(\phi_t)^*\big(\mathbb{B}(\mathsf{s},\nu)\big)d\mu$$
$$= \frac{d}{dt}\int_{U\Sigma} (\phi_t)^*\big(\mathbb{B}(\mathsf{s},\nu)\big)d\mu$$
$$= \frac{d}{dt}\int_{U\Sigma} \mathbb{B}(\mathsf{s},\nu)d\big((\phi_t)_*\mu\big) = 0.$$

The second term vanishes since $\nabla_\xi(\nu) = 0$ (Lemma 3.1). □

Thus

$$\Psi_{[\mathsf{u}]}(\mu) := \int_{U\Sigma} F_{\mathsf{u},s} d\mu$$

is a well-defined function $\mathcal{C}(\Sigma) \xrightarrow{\Psi_{[\mathsf{u}]}} \mathbb{R}$ whose continuity follows from the definition of the weak topology on $\mathcal{P}(U\Sigma)$.

5.2. **Periodic orbits.** Suppose that

$$t \longmapsto \Gamma_0 g a(t)$$

defines a periodic orbit of the geodesic flow ϕ_t. Suppose that γ is the corresponding element of Γ_0. Then

$$\Gamma_0 g a(t+T) = \Gamma_0 \gamma g a(t) = \Gamma_0 g a(t)$$

where $T = \ell(\gamma)$ is the period of the orbit.

The following notation is useful. If $x_0 \in U\Sigma$ and $T > 0$, let $\phi^{x_0}_{[0,T]}$ denote the map

$$\begin{aligned} [0,T] &\xrightarrow{\phi^{x_0}_{[0,T]}} U\Sigma \\ t &\longmapsto \phi_t(x_0). \end{aligned} \tag{5.1}$$

Let $T = \ell(\gamma)$ denote the period of the periodic orbit corresponding to γ and $\mu_{[0,T]}$ denote Lebesgue measure on $[0,T]$. Then

$$\mu_\gamma := \frac{1}{T}(\phi^{x_0}_{[0,T]})_*\big(\mu_{[0,T]}\big) \tag{5.2}$$

defines the *geodesic current associated to the periodic orbit* γ.

Proposition 5.2 (Labourie [27], Proposition 4.2)**.** *Let $\gamma \in \Gamma_0$ be hyperbolic and let $\mu_\gamma \in \mathcal{C}_{\mathsf{per}}(\Sigma)$ be the corresponding geodesic current. Then*

$$\alpha(\gamma) = \ell(\gamma) \int_{U\Sigma} F_{\mathsf{u},s}\, d\mu_\gamma. \tag{5.3}$$

Proof. Let $x_0 \in U\Sigma$ is a point on the periodic orbit, and consider the map (5.1) and the geodesic current (5.2).

Choose a section $s \in \mathcal{S}(\mathbb{E})$. The pullback of s to the covering space $U\mathsf{H}^2 \approx \mathsf{G}_0$ is the graph of a map

$$\tilde{v} : \mathsf{G}_0 \longrightarrow \mathsf{V}$$

satisfying

$$\tilde{v} \circ \gamma = \rho(\gamma)\tilde{v}.$$

Then

$$\int_{U\Sigma} F_{\mathsf{u},s}\, d\mu_\gamma = \frac{1}{T} \int_0^T F_{\mathsf{u},s}(\phi_t(x_0))dt, \tag{5.4}$$

which we evaluate by passing to the covering space $U\mathsf{H}^2 \approx \mathsf{G}_0$.

Lift the basepoint x_0 to $\tilde{x}_0 \in U\mathsf{H}^2$. Then $\phi^{x_0}_{[0,T]}$ lifts to the map

$$\begin{aligned} [0,T] &\xrightarrow{\tilde{\phi}^{\tilde{x}_0}_{[0,T]}} U\mathsf{H}^2 \\ t &\longmapsto \tilde{\phi}_t(\tilde{x}_0). \end{aligned}$$

Let $g \in \mathsf{G}_0$ correspond via $\mathcal{E}$ to $\tilde{x}_0$. Then the periodic orbit lifts to the trajectory

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathsf{G}_0 \\ t &\longmapsto ga(-t). \end{aligned}$$

Since the periodic orbit corresponds to the deck transformation γ (which acts by left-multiplication on G_0),

$$\gamma g = ga(-T)$$

which implies

$$\gamma = ga(-T)g^{-1}. \tag{5.5}$$

Evaluate $\nabla_\xi s$ and ν along the trajectory

$$\tilde{\phi}_t\tilde{x}_0 \longleftrightarrow ga(-t)$$

by lifting to the covering space and computing in G. Lift s to

$$\begin{aligned} \mathsf{G}_0 &\xrightarrow{\tilde{s}} V \rtimes \mathsf{G}_0 = G \\ g &\longmapsto (\tilde{\mathsf{v}}(g), g) \end{aligned}$$

where $\tilde{\mathsf{v}} : \mathsf{G}_0 \longrightarrow V$ satisfies

$$\tilde{\mathsf{v}}(\gamma g) = \rho(\gamma)\tilde{\mathsf{v}}(g).$$

Then

$$\big(\nabla_\xi \tilde{s}\big)(\phi_t \tilde{x}_0) = \frac{D}{dt}\tilde{\mathsf{v}}(ga(-t)).$$

In semidirect product coordinates, the lift $\tilde{\nu}$ is defined by the map

$$\begin{aligned} \mathsf{G}_0 &\xrightarrow{\tilde{\nu}} V \\ g &\longmapsto \mathbb{L}(g)\mathsf{v}_0. \end{aligned}$$

Lemma 5.3. *$\tilde{\nu}\big(ga(-t)\big) = \mathsf{x}^0(\gamma)$ for all $t \in \mathbb{R}$.*

Proof.

$$\begin{aligned} \tilde{\nu}\big(ga(-t)\big) &= \mathbb{L}\big(ga(-t)\big)\ \mathsf{v}_0 \\ &= \mathbb{L}\big(ga(-t)\big)\ \mathsf{x}^0\big(a(-T)\big) \\ &= \mathsf{x}^0\big((ga(-t))a(-T)(ga(-t))^{-1}\big) \\ &= \mathsf{x}^0\big((ga(-T)g^{-1}\big) = \mathsf{x}^0(\gamma) \qquad\qquad \text{(by (5.5))}. \end{aligned}$$

□

We evaluate the integrand in (5.4):

$$\begin{aligned} F_{\mathsf{u},s}(\phi_t x_0) &= \mathbb{B}(\nabla_\xi s, \nu)(\phi_t x_0) \\ &= \mathsf{B}\big(\nabla_\xi \tilde{s}(ga(-t)), \tilde{\nu}(ga(-t)\big) \\ &= \mathsf{B}\left(\frac{D}{dt}\tilde{\mathsf{v}}(ga(-t)), \mathsf{x}^0(\gamma)\right) \\ &= \frac{d}{dt}\mathsf{B}\big(\tilde{\mathsf{v}}(ga(-t)), \mathsf{x}^0(\gamma)\big) \end{aligned}$$

and

$$\begin{aligned} \int_0^T F_{\mathsf{u},s}(\phi_t x_0) dt &= \mathsf{B}\big(\tilde{\mathsf{v}}(ga(-T)), \mathsf{x}^0(\gamma)\big) - \mathsf{B}\big(\tilde{\mathsf{v}}(g), \mathsf{x}^0(\gamma)\big) \\ &= \mathsf{B}\big(\tilde{\mathsf{v}}(\gamma g) - \tilde{\mathsf{v}}(g), \mathsf{x}^0(\gamma)\big) \\ &= \mathsf{B}\big(\rho(\gamma)\tilde{\mathsf{v}}(g) - \tilde{\mathsf{v}}(g), \mathsf{x}^0(\gamma)\big) \\ &= \alpha(\gamma) \end{aligned}$$

as claimed. □

5.3. **Nonproper deformations.** Now we prove that $0 \notin \Psi_{[\mathsf{u}]}(\mathcal{C}(\Sigma))$ implies Γ_{u} acts properly.

Proposition 5.4. *Suppose that Φ defines a non-proper action. Then there exists a ϕ-invariant Borel probability measure μ on $U\Sigma$ such that $\Psi_{[\mathsf{u}]}(\mu) = 0$.*

Proof. By Proposition 4.3, we may assume that Φ defines a non-proper action on $\mathbb{E}$. There exist compact subsets $K_1, K_2 \subset \mathbb{E}$ for which the set of $t \in \mathbb{R}$ such that

$$\Phi_t(K_1) \cap K_2 \neq \emptyset$$

is noncompact. Thus there exists an unbounded sequence $t_n \in \mathbb{R}$, and sequences $P_n \in K_1$ and $Q_n \in K_2$ such that

$$\Phi_{t_n} P_n = Q_n.$$

Passing to subsequences, we may assume that $t_n \nearrow +\infty$ and

$$\lim_{n\to\infty} P_n = P_\infty, \qquad \lim_{n\to\infty} Q_n = Q_\infty. \tag{5.6}$$

for some $P_\infty, Q_\infty \in \mathbb{E}$. For $n = 1, \dots, \infty$, the images

$$p_n = \Pi(P_n),\ q_n = \Pi(Q_n)$$

are points in $U\Sigma$ such that

$$\lim_{n\to\infty} p_n = p_\infty, \qquad \lim_{n\to\infty} q_n = q_\infty \tag{5.7}$$

and $\phi_{t_n}(p_n) = q_n$. Choose $R > 0$ such that

$$d(p_\infty, U\mathsf{core}(\Sigma)) \ < \ R,$$
$$d(q_\infty, U\mathsf{core}(\Sigma)) \ < \ R.$$

Passing to a subsequence we may assume that all p_n, q_n lie in the compact set $U\mathcal{K}_R$. Since $\mathcal{K}_R$ is geodesically convex, the curves

$$\{\phi_t(p_n) \mid 0 \le t \le t_n\}$$

also lie in $U\mathcal{K}_R$(Lemma 4.1).

Choose a section $s \in \mathcal{S}(\mathbb{E})$. For $n = 1, \dots, \infty$, write

$$P_n = s(p_n) + a_n\nu + \mathsf{p}_n^+ + \mathsf{p}_n^-$$
$$Q_n = s(q_n) + b_n\nu + \mathsf{q}_n^+ + \mathsf{q}_n^- \tag{5.8}$$

where $a_n, b_n \in \mathbb{R}$, $\mathsf{p}_n^-, \mathsf{q}_n^- \in \mathbb{V}^-$, and $\mathsf{p}_n^+, \mathsf{q}_n^- \in \mathbb{V}^+$. Since

$$D\Phi_t(\nu) = \nu$$

and $\Phi_{t_n}(P_n) = Q_n$, taking $\mathbb{V}^0$-components in (5.8) yields:

$$\int_0^{t_n} (F_{\mathsf{u},s} \circ \phi_t)(p_n)\, dt \ = \ b_n - a_n. \tag{5.9}$$

Since s is continuous, (5.7) implies $s(p_n) \to s(p_\infty)$ and $s(q_n) \to s(q_\infty)$. By (5.6),

$$\lim_{n\longrightarrow\infty} (b_n - a_n) \;=\; b_\infty - a_\infty.$$

Thus (5.9) implies

$$\lim_{n\to\infty} \int_0^{t_n} (F_{\mathsf{u},s} \circ \phi_t)(p_n)\, dt \;=\; b_\infty - a_\infty$$

so

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (F_{\mathsf{u},s} \circ \phi_t)(p_n)\, dt \;=\; 0 \tag{5.10}$$

(because $t_n \nearrow +\infty$).

Define approximating probability measures

$$\mu_n \in \mathcal{P}(U\mathcal{K}_R)$$

by pushing forward Lebesgue measure on the orbit through p_n and dividing by t_n:

$$\mu_n := \frac{1}{t_n} (\phi^{p_n}_{[0,t_n]})_* \mu_{[0,t_n]}$$

where $\phi^{p_n}_{[0,t_n]}$ is defined in (5.1). Compactness of $U\mathcal{K}_R$ guarantees a weakly convergent subsequence μ_n in $\mathcal{P}(U\mathcal{K}_R)$. Thus

$$\Psi_{[\mathsf{u}]}(\mu_\infty) = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (F_{\mathsf{u},s} \circ \phi_t)(p_n) dt = 0$$

by (5.10).

μ_∞ is ϕ-invariant: For any $f \in L^\infty(U\mathcal{K}_R)$ and $\lambda \in \mathbb{R}$,

$$\left| \int f d\mu_n - \int (f \circ \phi_\lambda) d\mu_n \right| < \frac{2\lambda}{t_n} \|f\|_\infty$$

for n sufficiently large. Passing to the limit,

$$\left| \int f\, d\mu_\infty \;-\; \int (f \circ \phi_\lambda)\, d\mu_\infty \right| \;=\; 0.$$

$\square$

6. Properness

Now we prove that $\Psi_{[\mathsf{u}]}(\mu) \neq 0$ for a proper deformation Γ_{u}. The proof uses a lemma ensuring that the section s can be chosen only to vary in the neutral direction ν. The proof uses the hyperbolicity of the geodesic flow. The uniform positivity or negativity of $\Psi_{[\mathsf{u}]}$ implies Margulis's "Opposite Sign Lemma' (Margulis' [28, 29], Abels [1], Drumm [14, 16]) as a corollary.

Lemma 6.1. *There exists a section $s \in \mathcal{S}(\mathbb{E})$ such that $\nabla_\xi(s) \in \mathbb{V}^0$.*

Proof. Let $s \in \mathcal{S}(\mathbb{E})$. Decompose $\nabla_\xi(s)$ by the splitting (4.3) into components:

$$\nabla_\xi s = \nabla_\xi^- s + \nabla_\xi^0 s + \nabla_\xi^+ s.$$

where $\nabla_\xi^\pm(s) \in \mathbb{V}^\pm$. Since $D\Phi$ exponentially contracts V^- as $t \to +\infty$

$$\|D\Phi_t(\mathsf{v})\| \leq Ce^{-kt}\|\mathsf{v}\|$$

for $\mathsf{v} \in \mathsf{V}^-$. Then

$$\begin{aligned}\left\| \int_0^T D\Phi_t\big(\nabla_\xi^-(s(\phi_{-t}(p)))\big)dt \right\| &\leq \int_0^T \|D\Phi_t\big(\nabla_\xi^-(s(\phi_{-t}(p)))\big)\|dt \\ &\leq \frac{1-e^{-kT}}{k}\|\nabla_\xi^-(s)\|_\infty\end{aligned}$$

so

$$\int_0^\infty D\Phi_t^*\big(\nabla_\xi^-(s)\big)dt = \lim_{T\to\infty}\int_0^T D\Phi_t^*\big(\nabla_\xi^-(s)\big)dt$$

and the convergence is uniform on compact subsets. Similarly

$$\int_0^\infty \phi_{-t}^* \nabla_\xi^+(s)dt$$

converges uniformly on compact subsets. Thus

$$s - \int_0^\infty D\Phi_t^*\nabla_\xi^-(s)dt - \int_0^\infty D\Phi_{-t}^*\nabla_\xi^+(s)dt$$

is a section in $\mathcal{S}(\mathbb{E})$ whose covariant derivative with respect to ξ lies in $\mathbb{V}^0$, as claimed. □

Proposition 6.2. *Suppose that Φ defines a proper action. Then*

$$\Psi_{[\mathsf{u}]}(\mu) \neq 0$$

for all $\mu \in \mathcal{C}(\Sigma)$.

Proof. Let $\mu \in \mathcal{C}(\Sigma)$ and $p \in \operatorname{supp}(\mu) \subset U_{\mathsf{rec}}\Sigma$. Applying Lemma 6.1, choose a section s with $\nabla_\xi(s) \in \mathbb{V}^0$. Then

$$\Phi_T(s(p)) \;=\; s(\phi_T(p)) \;+\; \left(\int_0^T F_{\mathsf{u},s}(\phi_t(p))\,dt\right)\nu \tag{6.1}$$

Since Φ is proper, for every pair of compact subsets $K_1, K_2 \in \mathbb{E}$, the set of $t \in \mathbb{R}$ for which

$$\Phi_t(K_1) \cap K_2 \neq \emptyset$$

is compact. Take $K_1 = s(\mathrm{supp}(\mu))$. Thus $\Phi_t s(p) \to \infty$ uniformly for $p \in \mathrm{supp}(\mu)$, as $t \to \infty$. Compactness of $\mathrm{supp}(\mu)$ implies that $s(\phi_t p) - s(p)$ is bounded. Thus (6.1) implies

$$\int_0^T F_{\mathsf{u},s}(\phi_t p)\, dt \longrightarrow \infty$$

uniformly in $p \in \mathrm{supp}(\mu)$, as $T \longrightarrow \infty$. Furthermore, connectedness of α-limit sets and boundedness of $F_{\mathsf{u},s}$ implies that

$$\int_0^\infty F_{\mathsf{u},s}(\phi_t p)\, dt \;=\; +\infty.$$

Thus

$$\lim_{T\longrightarrow\infty} \int d\mu \int_0^T F_{\mathsf{u},s}(\phi_t p)\, dt \;=\; +\infty. \tag{6.2}$$

Fubini's theorem implies

$$\int d\mu(p) \int_0^T F_{\mathsf{u},s}(\phi_t p)\, dt \;=\; \int_0^T \left(\int F_{\mathsf{u},s}(\phi_t p) d\mu(p) \right) dt. \tag{6.3}$$

Invariance of μ implies

$$\int F_{\mathsf{u},s}(\phi_t p) d\mu(p) = \int F_{\mathsf{u},s} d\mu. \tag{6.4}$$

Thus (6.3) and (6.4) imply

$$\begin{aligned} \int d\mu \int_0^T F_{\mathsf{u},s}(\phi_t p) dt &= \int_0^T \left(\int F_{\mathsf{u},s}(\phi_t p) d\mu \right) dt \\ &= \int_0^T \left(\int F_{\mathsf{u},s} d\mu \right) dt \;=\; T \int F_{\mathsf{u},s} d\mu. \end{aligned} \tag{6.5}$$

Suppose that $\int F_{\mathsf{u},s} d\mu = 0$. Taking the limit in (6.5) as $T \longrightarrow \infty$ contradicts (6.2). Thus $\int F_{\mathsf{u},s} d\mu \neq 0$, as desired. $\square$

Corollary 6.3 (Margulis [28, 29])**.** *Suppose that $\gamma_1, \gamma_2 \in \Gamma_0$ satisfy*

$$\alpha(\gamma_1) < 0 < \alpha(\gamma_2).$$

Then Γ does not act properly.

Proof. Proposition 5.2 implies that

$$\Psi_{[\mathsf{u}]}(\mu_{\gamma_1}) < 0 < \Psi_{[\mathsf{u}]}(\mu_{\gamma_2})$$

Convexity of $\mathcal{C}(\Sigma)$ implies a continuous path $\mu_t \in \mathcal{C}(\Sigma)$ exists, with $t \in [1, 2]$, for which

$$\mu_1 = \mu_{\gamma_1},$$
$$\mu_2 = \mu_{\gamma_2}.$$

The function

$$\mathcal{C}(\Sigma) \xrightarrow{\Psi_{[\mathsf{u}]}} \mathbb{R}$$

is continuous. The intermediate value theorem implies $\Psi_{[\mathsf{u}]}(\mu_t) = 0$ for some $1 < t < 2$. Proposition 6.2 implies that Γ does not act properly. $\square$

References

1. Abels, H., *Properly discontinuous groups of affine transformations, A survey,* Geometriae Dedicata **87** (2001) 309–333.
2. ______, Margulis, G., and Soifer, G., *Properly discontinuous groups of affine transformations with orthogonal linear part,* C. R. Acad. Sci. Paris Sr. I Math. **324** (3) (1997), 253–258.
3. ______, *On the Zariski closure of the linear part of a properly discontinuous group of affine transformations,* J. Diff. Geo. **60** (2) (2002), 315–344.
4. Bedford, T., Keane, M., Series, C., "Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces," Oxford Science Publications, Oxford University Press (1991).
5. Benoist, Y., *Actions propres sur les espaces homogènes réductifs,* Ann. Math. **144** (1996), 315–347.
6. Bonahon, F., *The geometry of Teichmüller space via geodesic currents,* Invent. Math. **92** (1988), 139–162.
7. Bourbaki, N., "Topologie Générale", Éléments de Mathématique, Hermann, Paris (1966).
8. Canary, R., Epstein, D. B. A., and Green, P. *Notes on notes of Thurston,* in "Analytical and Geometric Aspects of Hyperbolic Space," London Math. Soc. Lecture Notes **111,** Cambridge University Press, London (1987), 3–92.
9. Charette, V., *Non-proper actions of the fundamental group of a punctured torus,* (submitted).
10. ______ and Drumm, T., *Strong marked isospectrality of affine Lorentzian groups,* J. Diff. Geo. (to appear).
11. Charette, V. and Goldman, W., *Affine Schottky groups and crooked tilings,* in "Crystallographic Groups and their Generalizations," Contemp. Math. **262** (2000), 69–97, Amer. Math. Soc.
12. Drumm, T., *Fundamental polyhedra for Margulis space-times,* Doctoral Dissertation, University of Maryland (1990).
13. ______, *Fundamental polyhedra for Margulis space-times,* Topology **31** (4) (1992), 677-683.
14. ______, *Examples of nonproper affine actions,* Mich. Math. J. **30** (1992), 435–442.
15. ______, *Linear holonomy of Margulis space-times,* J. Diff. Geo. **38** (1993), 679–691.
16. ______, *Translations and the holonomy of complete affine manifolds,* Math. Res. Lett. **1** (1994), 757–764.
17. ______ and Goldman, W., *Complete flat Lorentz 3-manifolds with free fundamental group,* Int. J. Math. **1** (1990), 149–161.
18. ______, *Isospectrality of flat Lorentz 3-manifolds,* J. Diff. Geo. **58** (3) (2001), 457–466.

19. Fried, D. and Goldman, W. , *Three-dimensional affine crystallographic groups,* Adv. Math. **47** (1983), 1–49.
20. Goldman, W., *The Margulis Invariant of Isometric Actions on Minkowski (2+1)-Space,* in "Ergodic Theory, Geometric Rigidity and Number Theory," Springer-Verlag (2002), 149–164.
21. ——— and Hirsch, M., *The radiance obstruction and parallel forms on affine manifolds,* Trans. Amer. Math. Soc. **286** (1984), 639–649.
22. Goldman, W. and Margulis, G., *Flat Lorentz 3-manifolds and cocompact Fuchsian groups,* in "Crystallographic Groups and their Generalizations," Contemp. Math. **262** (2000), 135—146, Amer. Math. Soc.
23. Jones, C., *Pyramids of properness: Towards the Properness Conjecture,* Doctoral Dissertation, University of Maryland (2003).
24. Kapovich, M., "Hyperbolic Manifolds and Discrete Groups," Progress in Math. **183** (2001) Birkhäuser Boston.
25. Kobayashi, S. and Nomizu, K., "Foundations of Differential Geometry," Volume 1, Interscience Tracts in Pure and Applied Mathematics **15** (1963), John Wiley & Sons, New York—London.
26. Koszul, J. L., *Lectures on Groups of Transformations,* Tata Institute of Fundamental Research, Bombay (1965).
27. Labourie, F., *Fuchsian affine actions of surface groups,* J. Diff. Geo. **59** (1), (2001), 15 – 31.
28. Margulis, G., *Free properly discontinuous groups of affine transformations,* Dokl. Akad. Nauk SSSR **272** (1983), 937–940.
29. ———, *Complete affine locally flat manifolds with a free fundamental group,* J. Soviet Math. **134** (1987), 129–134.
30. Mess, G., *Lorentz spacetimes of constant curvature,* I. H. E. S. preprint (1990).
31. Milnor, J., *On fundamental groups of complete affinely flat manifolds,* Adv. Math. **25** (1977), 178–187.
32. Palais, R., *On the existence of slices for actions of non-compact Lie groups,* Ann. Math. 73 No. 2, (1961), 295–323.
33. Rieffel, M., *Strong Morita equivalence of certain transformation group C^*-algebras,* Math. Ann. **222** (1), (1976), 7–22.
34. ———, *Applications of strong Morita equivalence to transformation group C^*-algebras,* in "Operator Algebras and applications, Part I (Kingston, Ont., 1980)," Proc. Symp. Pure Math. **38**, Amer. Math. Soc. (1982), 299–310.
35. Series, C., *Geometrical Markov coding of geodesics on surfaces of constant negative curvature,* Ergod. Th. and Dynam. Sys. **6** (1986), 601–625.
36. Tits, J., *Free subgroups in linear groups,* J. Algebra **20** (1972), 250–270.

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