ENERGY OF TWISTED HARMONIC MAPS OF RIEMANN SURFACES

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ABSTRACT. The energy of harmonic sections of flat bundles of nonpositively curved (NPC) length spaces over Riemann surfaces is a function on Teichmüller space which is a qualitative invariant of the holonomy representation. Adapting ideas of Sacks-Uhlenbeck, Schoen-Yau and Tromba, we show that the energy function is proper for any convex cocompact representation of the fundamental group. As a corollary, the mapping class group acts properly on the subset of convex cocompact representations.

CONTENTS

INTRODUCTION

Let S be a closed oriented surface with $\chi(S) < 0$, and let $\pi_1(S)$ be its fundamental group. Let (X, d) be a complete nonpositively curved length space with isometry group G. In this note we discuss an invariant of isometric actions of $\pi_1(S)$ on X.

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Choose a universal covering space $\widetilde{S} \longrightarrow S$ with group of deck transformations $\pi_1(S)$. An isometric action of $\pi_1(S)$ on X is a homomorphism $\pi_1(S) \xrightarrow{\rho} G$. Such a homomorphism defines a *flat* (G, X)*-bundle* X_ρ over S, whose total space is the quotient $\widetilde{S} \times X$ by the (diagonal) π-action by deck transformations on $\widetilde{S}$ and by ρ on X. Since X is contractible (see Bridson-Haefliger [4]), sections always exist and correspond to ρ-equivariant mappings $\widetilde{S} \xrightarrow{u} X$.

For us, a *conformal structure* on S will be an *almost complex structures*σ on S, that is, an automorphism of the tangent bundle TS satisfying $\sigma^2 = -\mathbf{I}$. A Riemannian metric g is *in the conformal class of* σ if and only if $g(\sigma v_1, \sigma v_2) = g(v_1, v_2)$ for tangent vectors v_1, v_2.

Choose a conformal structure σ on S, and a Riemannian metric g in the conformal class of σ. For every ρ-equivariant mapping $(\widetilde{S}, g) \xrightarrow{f} X$, its *energy* $E_{\rho,g}(f)$ is defined as follows. Define the *energy density* on $\widetilde{S}$ as

$$\|df\|^2 \, dA,$$

where $\|\|$ denotes the Hilbert-Schmidt norm with respect to the Riemannian metrics on $\widetilde{S}$ and X and dA the Riemannian area form on $\widetilde{S}$. The energy density on $\widetilde{S}$ is $\pi_1(S)$-invariant and hence defines an exterior 2-form, the *energy density* on S. Define $E_{\rho,g}(f)$ as its integral over S. Alternatively, $E_{\rho,g}(f)$ is the integral of the energy density on $\widetilde{S}$ over a fundamental domain for the $\pi_1(S)$-action on $\widetilde{S}$. Since S is two-dimensional, $E_{\rho,g}(f)$ (if finite) depends only on the conformal structure σ, and we denote it $E_{\rho,\sigma}(f)$. Finite energy maps always exist, and [14, Theorem 2.6.4] guarantees an energy minimizing sequence of uniformly Lipschitz equivariant mappings.

The identity component $\mathsf{Diff}^0(S)$ of the group $\mathsf{Diff}(S)$ of all diffeomorphisms acts on conformal structures, with orbit space the *Teichmüller space* $\mathcal{T}_S$ of S.

Fix the surface S and a representation ρ of its fundamental group. Letting σ vary over all conformal structures on S, the representation ρ determines a function

$$\begin{aligned} \mathcal{T}_S &\xrightarrow{E_\rho} \mathbb{R} \\ [\sigma] &\longmapsto \inf_f \, E_{\sigma,\rho}(f). \end{aligned}$$

Its qualitative properties are invariants of the representation ρ. For example, E_ρ is constant if and only if the image of ρ is precompact.

Perhaps the most interesting situation is when the infimum of the energy is realized. In the context of NPC targets, recall that a map f

is called *harmonic* if it minimizes $E_{\rho,\sigma}$ among all ρ equivariant maps of finite energy. While a minimizer may fail to be unique, clearly $E_{\rho,\sigma} = E_{\rho,\sigma}(f)$ is independent of the choice of harmonic map and is thus an invariant of the pair (ρ, σ). If ρ is *reductive*, then by Corlette [6], Donaldson [7], Eells-Sampson [10], Labourie [16], Korevaar-Schoen[14], such a harmonic map exists. **We should define and discuss reductivity.**

We consider a somewhat different class of representations. Recall that a discrete subgroup $\Gamma \subset G$ is *convex cocompact* if there exists a Γ-invariant closed convex subset $N \subset X$ such that N/Γ is compact. A representation ρ is *convex cocompact* if ρ is an isomorphism of $\pi_1(S)$ onto a convex cocompact discrete subgroup of G. In Proposition 2.1 below, we show that equivariant harmonic maps always exist for convex cocompact representations.

Theorem A. *Suppose that ρ is convex cocompact. Then E_ρ is a proper function on $\mathcal{T}_S$.*

The basic idea goes back to work of Sacks-Uhlenbeck [22] and Schoen-Yau [24]. When ρ is a Fuchsian representation (corresponding to a hyperbolic structure on S), Tromba [27] proved that E_ρ is proper and has a unique critical point (necessarily a minimum). When ρ is a quasi-Fuchsian $\mathsf{PSL}(2,\mathbb{C})$-representation, E_ρ is proper. Uhlenbeck [28] gave an explicit criterion for when E_ρ has a unique minimum. Generally E_ρ admits more than one critical point, for quasi-Fuchsian ρ.

One motivation for this result is the following. The *mapping class group* $\pi_0\mathsf{Diff}^+(S)$ acts on the space $\mathrm{Hom}(\pi_1(S), G)/G$. Let $\mathfrak{C}$ be the subset of $\mathrm{Hom}(\pi_1(S), G)/G$ consisting of equivalence classes of convex cocompact representations.

Corollary B. *$\pi_0\mathsf{Diff}^+(S)$ acts properly on $\mathfrak{C}$.*

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Notation and Terminology

If X is a metric space, we denote the distance function by d_X. If c is a curve in a length space X, we denote its length by $L_X(c)$. We denote by $[a]$ the equivalence class of a, in various contexts. We denote the identity transformation by $\mathbf{I}$. We shall sometimes implicitly assume a fixed basepoint in discussing the fundamental group $\pi_1(S)$ and the corresponding universal covering space $\widetilde{S} \longrightarrow S$.

1. Bounded geometry

Let $\gamma \in G$. Its translation length $|\gamma|$ is defined by:

$$|\gamma| := \inf_{x \in X} d(x, \gamma x) \ . \tag{1.1}$$

Lemma 1.1. *Let $\Gamma \subset G$ be a convex cocompact discrete subgroup. Then $\exists \varepsilon_0 > 0$ such that $|\gamma| \geq \varepsilon_0$ for all $\gamma \in \Gamma \setminus \{\mathbf{I}\}$.*

Proof. Suppose not. Then $\exists \gamma_i \in \Gamma$ such that $|\gamma_i| \neq 0$ for all i, and $|\gamma_i| \to 0$. Let N be a closed convex Γ-invariant subset such that N/Γ is compact. Since N is convex and Γ-invariant, $\exists x_i \in N$ such that

$$d(x_i, \gamma_i x_i) \longrightarrow 0 \ .$$

Since N/Γ is compact, $\exists \lambda_i \in \Gamma$ and $x \in N$ such that, after passing to a subsequence, $\lambda_i x_i \to x$. Set $\tilde{\gamma}_i = \lambda_i \gamma_i \lambda_i^{-1}$. Then

$$d(\lambda_i x_i, \tilde{\gamma}_i \lambda x_i) \longrightarrow 0 \ .$$

Properness of the action of Γ near $x \in N$ implies that for only finitely many i does $|\tilde{\gamma}_i| = |\gamma_i|$ This contradicts the assumption that

$$0 \neq |\gamma_i| \to 0.$$

□

Lemma 1.2. *Let ε_0 satisfy Lemma 1.1. Let $\gamma_1, \gamma_2 \in \Gamma$ and $x, y \in X$. If*

- *$d(x, y) < \varepsilon_0/2$;*
- *$d(\gamma_1 x, \gamma_2 y) < \varepsilon_0/2$,*

then $\gamma_1 = \gamma_2$.

Proof.

$$\begin{aligned} |\gamma_2^{-1}\gamma_1| &\leq d(\gamma_2^{-1}\gamma_1 x, x) \\ &= d(\gamma_1 x, \gamma_2 x) \\ &\leq d(\gamma_1 x, \gamma_2 y) + d(\gamma_2 y, \gamma_2 x) \\ &= d(\gamma_1 x, \gamma_2 y) + d(x, y) \ < \ \varepsilon_0 \ . \end{aligned}$$

Now apply Lemma 1.1. □

2. Existence of harmonic maps

Proposition 2.1. *Suppose that $\pi_1(S) \xrightarrow{\rho} G$ is convex cocompact. Then there exists a ρ-equivariant harmonic map $\widetilde{S} \xrightarrow{u} X$.*

Proof. For any NPC space X and compact surface S, there exists an *energy minimizing* sequence u_i of uniformly Lipschitz ρ-equivariant mappings $\widetilde{S} \longrightarrow X$ (Korevaar-Schoen [14, Theorem 2.6.4]). Let $N \subset X$ be a ρ-invariant convex set such that $N/\rho(\pi_1(S))$ is compact. Projection $X \longrightarrow N$ decreases distances, and therefore decreases energy. Thus we may assume that the image of u_i lies in N.

Fix any point $\tilde{s} \in \widetilde{S}$. Since $N/\rho(\pi_1(S))$ is compact, after passing to a subsequence, $\exists c_i \in \pi_1(S)$ such that $v_i(\tilde{s})$ converges to a point in N, where

$$v_i := \rho(c_i) \circ u_i.$$

The v_i are uniformly Lipschitz and $v_i(\tilde{s})$ converges. The Arzéla-Ascoli theorem implies that a subsequence of v_i converges uniformly on compact subsets of $\widetilde{S}$. Choose $\varepsilon_0 > 0$ satisfying Lemma 1.1. For each compact $K \subset \widetilde{S}$, there exists $I > 0$ so that

$$d(v_i(w), v_j(w)) < \varepsilon_0/2 \tag{2.1}$$

whenever $i, j \geq I$ and $w \in K$.

Each v_i is equivariant with respect to $\rho_i = \rho \circ \mathbf{Inn}_{c_i}$, where $\mathbf{Inn}_{c_i}$ denotes the inner automorphism of $\pi_1(S)$ defined by c_i. Fix $i, j \geq I$, and set $x = v_i(\tilde{s})$ and $y = v_j(\tilde{s})$. Choose a finite generating set $\Pi \subset \pi_1(S)$. Applying (2.1) to the finite set $K = \Pi\tilde{s}$,

$$d(\rho_i(c)x, \rho_j(c)y) = d(v_i(c\tilde{s}), v_j(c\tilde{s})) < \varepsilon_0/2$$

whenever $c \in \Pi$. Since

$$d(x, y) = d(v_i(\tilde{s}), v_j(\tilde{s})) < \varepsilon_0/2 \ ,$$

Lemma 1.2 implies $\rho_i(c) = \rho_j(c)$ for all $c \in \Pi$. As Π generates $\pi_1(S)$ it follows $\rho_i = \rho_j$ if $i, j \geq I$. Since ρ is injective and $\pi_1(S)$ has trivial center, $c_i = c_j$ for all $i, j \geq I$. Therefore u_i itself converges locally uniformly to the desired minimizer. ☐

3. Properness of the Energy Function

We now prove that for ρ convex cocompact, the function E_ρ on $\mathcal{T}_S$ is proper. Suppose that $[\sigma_i] \in \mathcal{T}_S$ is a divergent sequence with

$$E_\rho([\sigma_i]) \leq B$$

for some constant $B > 0$, and all $i = 1, 2, \ldots$.

Assume that $d_{\mathcal{T}}([\sigma_i],[\sigma_j]) \geq 1$ for $i \neq j$, where $d_{\mathcal{T}}$ is the Teichmüller metric on $\mathcal{T}_S$. By [14] the ρ-equivariant harmonic maps

$$(\widetilde{S}, \widetilde{g_i}) \xrightarrow{u_i} X$$

have a uniform Lipschitz constant K, where $\widetilde{g_i}$ denotes the hyperbolic metric on $\widetilde{S}$ associated to σ_i. In particular, given a closed curve c in S, choose a lift $\tilde{c} \subset \widetilde{S}$ running from $\tilde{p}$ to $[c]\tilde{p}$, where $[c] \in \pi_1(S;p)$ is the deck transformation corresponding to c. We denote the length of c with respect to the metric g_i on S by $L_S(c)$. Then

$$\begin{aligned} |\rho([c])| &\leq d(u_i(\tilde{p}), \rho([c])u_i(\tilde{p})) \\ &= d(u_i(\tilde{p}), u_i(\rho([c])\tilde{p})) \\ &\leq L_X(u_i(\tilde{c})) \\ &\leq K L_S(c)\ . \end{aligned} \tag{3.1}$$

Suppose that $c \subset \Sigma$ is any closed essential curve. Since ρ is injective, the isometry $\rho(c)$ is nontrivial. Let $\varepsilon_0 > 0$ satisfy Lemma 1.1. Then (3.1) implies

$$\ell_c(\sigma_i) \geq \varepsilon_0/K$$

where $\ell_c(\sigma)$ denotes the *geodesic length function of* c *with respect to* σ, that is, the length of the unique closed geodesic freely homotopic to c in the hyperbolic metric corresponding to σ.

Mumford's compactness theorem [18] implies that the conformal structures σ_i lie in a compact subset of *moduli space* $\mathcal{T}_S/\pi_0\mathsf{Diff}^+(S)$. Thus $[\varphi_i] \in \pi_0\mathsf{Diff}^+(S)$ and $[\sigma_\infty] \in \mathcal{T}_S$ exist such that, after passing to a subsequence,

$$[\varphi_i][\sigma_i] \longrightarrow [\sigma_\infty]. \tag{3.2}$$

As $\mathsf{Diff}(S)$ acts properly on the set of Riemannian metrics [9, 8, 27], representatives g_i exist with $\varphi_i(g_i) \longrightarrow g_\infty$, where g_∞ denotes the hyperbolic metric associated to σ_∞.

Let $v_i = u_i \circ \varphi_i^{-1}$. The v_i are uniformly Lipschitz with respect to the metric g_∞ on S. In particular the family $\{v_i\}$ is equicontinuous. Choose a base point $\tilde{s} \in \widetilde{S}$. Since N/Γ is compact, $\exists \gamma_i \in \Gamma$ such that all $\gamma_i v_i(\tilde{s})$ lie in a compact subset of X. Since $\gamma_i = \rho(c_i)$ for some $c_i \in \pi_1(S)$, then (after possibly redefining φ_i), we may assume that $v_i(\tilde{s})$ itself is bounded. The Arzéla-Ascoli theorem then implies that a subsequence of v_i converges to v_∞ uniformly on compact sets. For I sufficiently large,

$$\sup_{z \in \widetilde{S}} d_X(v_i(z), v_j(z)) \ < \ \varepsilon_0/2 \tag{3.3}$$

for $i, j \geq I$.

Each v_i is equivariant with respect to $\rho_i = \rho \circ (\varphi_i)_*^{-1}$. Fix $i, j \geq I$, and let $x = v_i(\tilde{s})$ and $y = v_j(\tilde{s})$. For every $g \in \pi_1(S)$, (3.3) implies

$$\begin{aligned} d_X(\rho_i(g)x, \rho_j(g)y) &= d_X(v_i(g\tilde{s}), v_j(g\tilde{s})) \\ &< \varepsilon_0/2 \ . \end{aligned}$$

Since

$$d_X(x, y) = d_X(v_i(\tilde{s}), v_j(\tilde{s})) \ < \ \varepsilon_0/2 \ ,$$

Lemma 1.2 implies $\rho_i(\gamma) = \rho_j(\gamma)$ for all $\gamma \in \pi_1(S)$. Since ρ is injective, $(\varphi_i)_* = (\varphi_j)_*$.

The Dehn-Nielsen theorem (see Nielsen [20], Stillwell [25], Farb-Margalit [11]) implies the natural homomorphism

$$\pi_0 \mathsf{Diff}^+(S) \cong \mathrm{Out}(\pi_1(S)). \tag{3.4}$$

is an isomorphism. In particular, it is injective, so φ_i is isotopic to φ_j for all $i, j \geq I$. Call this common mapping class $[\varphi]$. Then for $i, j \geq I$,

$$\begin{aligned} d_{\mathcal{T}}([\sigma_i], [\sigma_j]) &= d_{\mathcal{T}}([\varphi][\sigma_i], [\varphi][\sigma_j]) \\ &= d_{\mathcal{T}}([\varphi_i][\sigma_i], [\varphi_j][\sigma_j]) \longrightarrow 0, \end{aligned}$$

by (3.2), contradicting the assumption that $d_{\mathcal{T}}([\sigma_i], [\sigma_j]) \geq 1$ for $i \neq j$.

Thus E_ρ is proper, as claimed.

4. Action of the mapping class group

Corollary B follows from the properness of the action of $\pi_0 \mathsf{Diff}^+(S)$ on $\mathcal{T}_S$ and a general fact on proper actions on metric spaces. Let X be a metric space and let $\mathcal{K}(X)$ denote the space of compact subsets of X, with the Hausdorff metric.

Lemma 4.1. *A group Γ of homeomorphisms of X acts properly on X if and only if Γ acts properly on $\mathcal{K}(X)$.*

Proof. The mapping

$$\begin{aligned} X &\xrightarrow{\iota} \mathcal{K}(X) \\ x &\longmapsto \{x\} \end{aligned}$$

is a proper isometric Γ-equivariant embedding. If Γ acts properly on $\mathcal{K}(X)$, then equivariance implies that Γ acts properly on X.

Conversely, suppose that Γ acts properly on X. For any compact subset $K \subset \mathcal{K}(X)$ of $\mathcal{K}(X)$, its union

$$UK := \bigcup_{A \in K} A$$

is a compact subset of X. For $\gamma \in \Gamma$ the condition

$$\gamma(K) \cap K \neq \emptyset \tag{4.1}$$

implies the condition

$$\gamma(UK) \cap UK \neq \emptyset. \tag{4.2}$$

To show that Γ acts properly on $\mathcal{K}(X)$, let $K \subset \mathcal{K}(X)$ be a compact subset. Since Γ acts properly on X, only finitely many $\gamma \in \Gamma$ satisfy (4.2), and hence only finitely many $\gamma \in \Gamma$ satisfy (4.1). Thus Γ acts properly on $\mathcal{K}(X)$. □

We now prove Corollary B. Let $[\rho] \in \mathfrak{C}$. By Theorem A, E_ρ is a proper function on $\mathcal{T}_S$, and assumes a minimum $m_0(E_\rho)$ Furthermore

$$\mathsf{Min}(\rho) := \{[\sigma] \in \mathcal{T}_S \mid E_\rho(\sigma) = m_0(E_\rho)\}$$

is a compact subset of $\mathcal{T}_S$, and

$$\mathfrak{C} \xrightarrow{\mathsf{Min}} \mathcal{K}(\mathcal{T}_S)$$

is a $\pi_0\mathsf{Diff}^+(S)$-equivariant continuous mapping.

The following classical fact is the basis for our proof:

Lemma 4.2. $\pi_0\mathsf{Diff}^+(S)$ *acts properly on* $\mathcal{T}_S$.

This theorem is commonly attributed to Fricke. Fricke's original proof uses the fact that $\pi_0\mathsf{Diff}^+(S)$ acts isometrically on $\mathcal{T}_S$ with respect to a metric (say, the Teichmüller metric), and that the marked length spectrum is discrete. Many good expositions of these ideas exist. See Abikoff [1], §2.2, Farb-Margalit [11], Harvey [12], §2.4.1, or Buser [5], §6.5.6 (p.156), Imayoshi-Tanigawa [13],§6.3, Nag [19],§2.7, Theorem II of Bers-Gardiner [3] for this proof.

Another proof follows from the properness of the diffeomorphism group $\mathsf{Diff}(S)$ of an arbitrary smooth manifold S on the space $\mathsf{Met}(S)$ of Riemannian metrics on S. This properness is due to Ebin [9] and Palais; see also Tromba [27]. In dimension two the uniformization theorem identifies $\mathcal{T}_S$ with the quotient of the $\mathsf{Diff}(S)$-invariant subset $\mathsf{Met}_{-1}(S)$ of $\mathsf{Met}(S)$ comprising metrics of curvature -1. Since $\mathsf{Diff}(S)$ acts properly on $\mathsf{Met}(S)$, it acts properly on $\mathsf{Met}_{-1}(S)$. Thus $\pi_0\mathsf{Diff}^+(S) = \mathsf{Diff}(S)/\mathsf{Diff}^0(S)$ acts properly on $\mathsf{Met}_{-1}(S)/\mathsf{Diff}^0(S)$, which identifies with $\mathcal{T}_S$. Still another proof, using closely related ideas, may be found in Earle-Eels [8].

Conclusion of Proof of Corollary B . Lemmas 4.2 and 4.1 together imply $\pi_0\mathsf{Diff}^+(S)$ acts properly on $\mathcal{K}(\mathcal{T}_S)$. By equivariance, $\pi_0\mathsf{Diff}^+(S)$ acts properly on $\mathfrak{C}$. □

When $G = \mathsf{PSL}(2,\mathbb{C})$, this is just the fact that $\pi_0\mathsf{Diff}^+(S)$ acts properly on the space $\mathcal{QF}_S$ of quasi-Fuchsian embeddings, which immediately follows from Bers's simultaneous uniformization theorem [2]. Namely, Bers constructed a Γ-equivariant homeomorphism

$$\mathcal{QF}_S \longrightarrow \mathcal{T}_S \times \bar{\mathcal{T}}_S.$$

Properness of the action of Γ on $\mathcal{T}_S$ implies properness on on $\mathcal{QF}_S$.

5. Kleiner's theorem

A natural generalization of Theorem A was suggested to us by Bruce Kleiner.

For general discrete embeddings of surface groups, E_ρ is not a proper map. If M^3 is a closed hyperbolic 3-manifold which fibers over the circle with fiber $S \subset M$, then the representation $\pi_1(S) \xrightarrow{\rho} \mathsf{PSL}(2,\mathbb{C})$ induced by the inclusion $S \hookrightarrow M$ and the holonomy representation $\pi_1(M) \hookrightarrow \mathsf{PSL}(2,\mathbb{C})$ will have a *normalizer* corresponding to the fibration. Specifically, let ϕ be an element of $\pi_1(M)$ which maps to a generator under the homomorphism $\pi_1(M) \longrightarrow \mathbb{Z}$ induced by the fibration. Let Φ denote the automorphism of $\pi_1(S)$ induced by $\mathbf{Inn}_\phi$. Let $g \in \mathsf{PSL}(2,\mathbb{C})$ denote the image of ϕ under the holonomy homomorphism $\pi_1(M) \longrightarrow \mathsf{PSL}(2,\mathbb{C})$. Then

$$\rho \circ \Phi = \mathbf{Inn}_g \circ \rho. \tag{5.1}$$

According to Thurston [26] (see also Otal [21]), Φ is a *pseudo-Anosov* or *hyperbolic* mapping class, and generates a proper $\mathbb{Z}$-action on $\mathcal{T}_S$. In particular every orbit is an infinite discrete subset of $\mathcal{T}_S$.

Since g is an isometry, (5.1) implies E_ρ is invariant under the action of Φ on $\mathcal{T}_S$. Since every Φ-orbit is discrete and infinite, E_ρ cannot be proper.

Theorem 5.1 (Kleiner)**.** *The function $\mathcal{T}_S/\langle\Phi\rangle \longrightarrow \mathbb{R}$ induced by E_ρ is proper.*

We deduce this result from the following generalization of Theorem A Let Γ be a discrete subgroup of G with a normal subgroup $\Gamma_1 \subset \Gamma$ isomorphic to the fundamental group of a closed surface S. Let $\pi_1(S) \xrightarrow{\rho} G$ denote composition of an isomorphism $\pi_1(S) \longrightarrow \Gamma_1$ with the inclusion $\Gamma_1 \hookrightarrow \Gamma \hookrightarrow G$. If Γ is convex cocompact, then a homotopy equivalence $S \longrightarrow X/\Gamma_1$ exists, which is *harmonic.* Furthermore the quotient group $Q := \Gamma/\Gamma_1$ acts on $\mathcal{T}_S$ preserving the function $\mathcal{T}_S \xrightarrow{E_\rho} \mathbb{R}$, and induces a function

$$\mathcal{T}_S/Q \xrightarrow{\hat{E}_\rho} \mathbb{R}$$

which is proper.

Proposition 5.2. *Suppose that* $\pi_1(S) \xrightarrow{\rho} G$ *is an isomorphism onto a normal subgroup* Γ_1 *of a convex cocompact group* Γ*. For every conformal structure* σ *on* S*, there exists a* ρ*-equivariant harmonic map* $(\widetilde{S}, \sigma) \xrightarrow{u} X$*.*

Proof. The proof follows exactly as in the proof of Proposition 2.1, except that now the representations $\rho_i = \rho \circ \widehat{\psi}(\gamma_i)$, where $\widehat{\psi} : \Gamma \to \mathrm{Aut}(\pi_1(S))$. By the assumption, $\widehat{\psi}$ is injective, and the remainder of the argument goes through as before. □

We continue with the notation as above. By the Dehn-Nielsen theorem (3.4), the image $\mathrm{im}(\psi)$ acts on $\mathcal{T}_S$. Note that E_ρ is invariant with respect to $\mathrm{im}(\psi)$.

6. Accidental parabolics

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